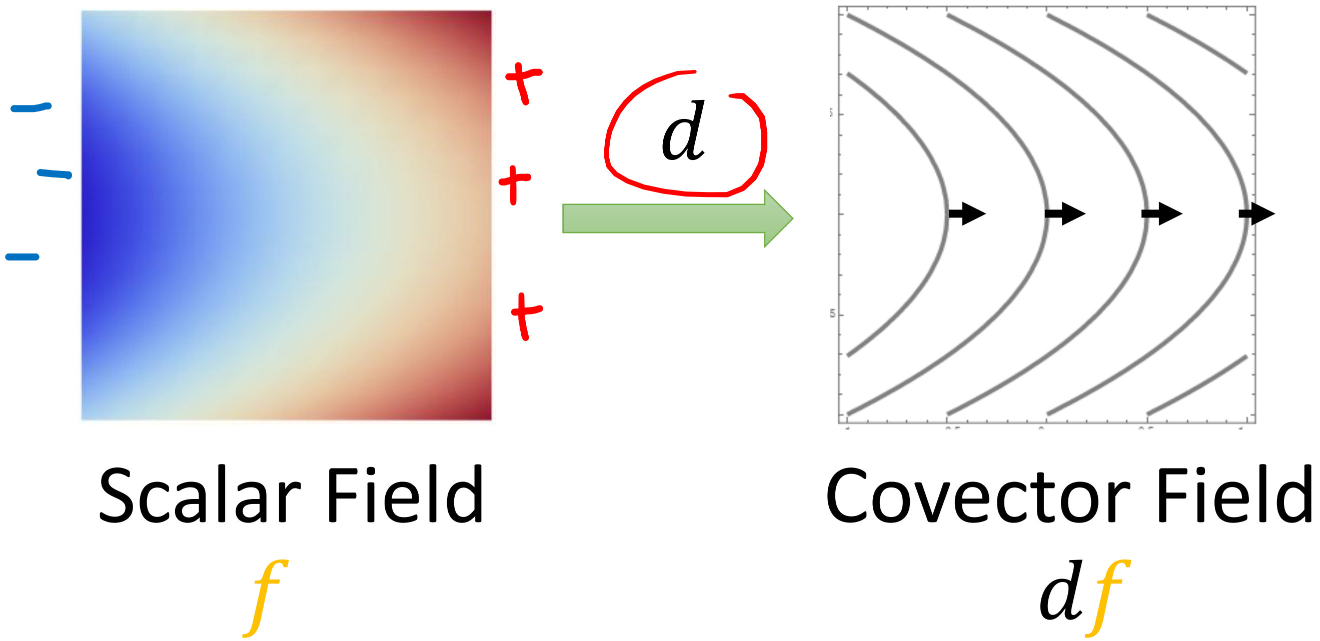
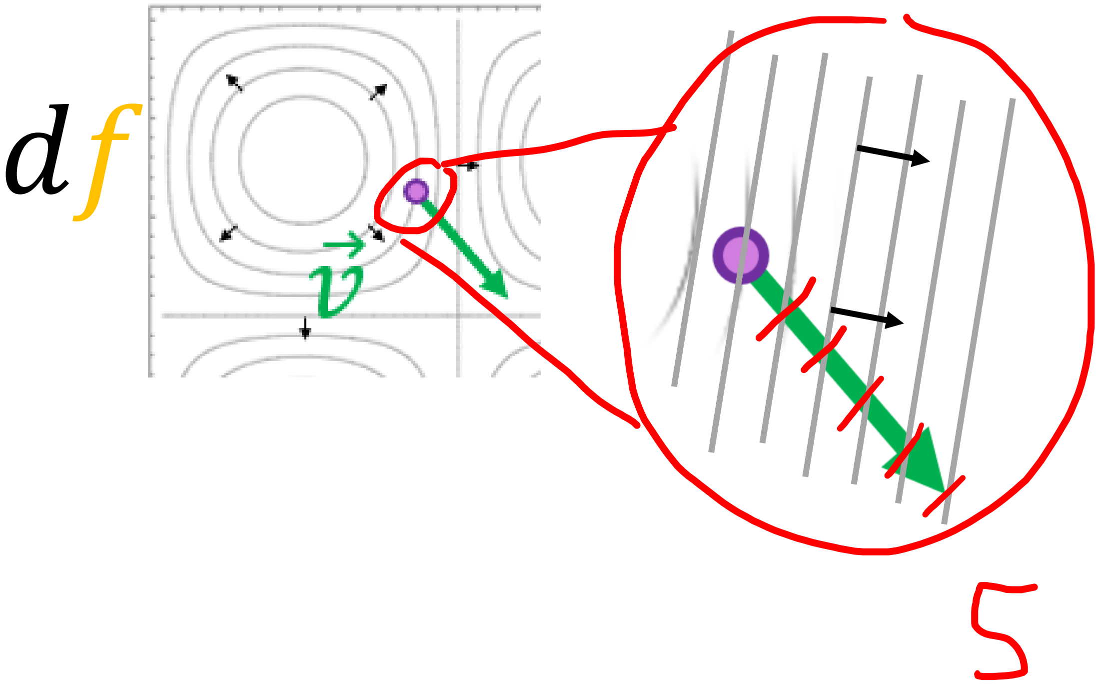
Integration With Differential Forms

(see links in description to learn about differential forms/covector fields)

Differential Forms

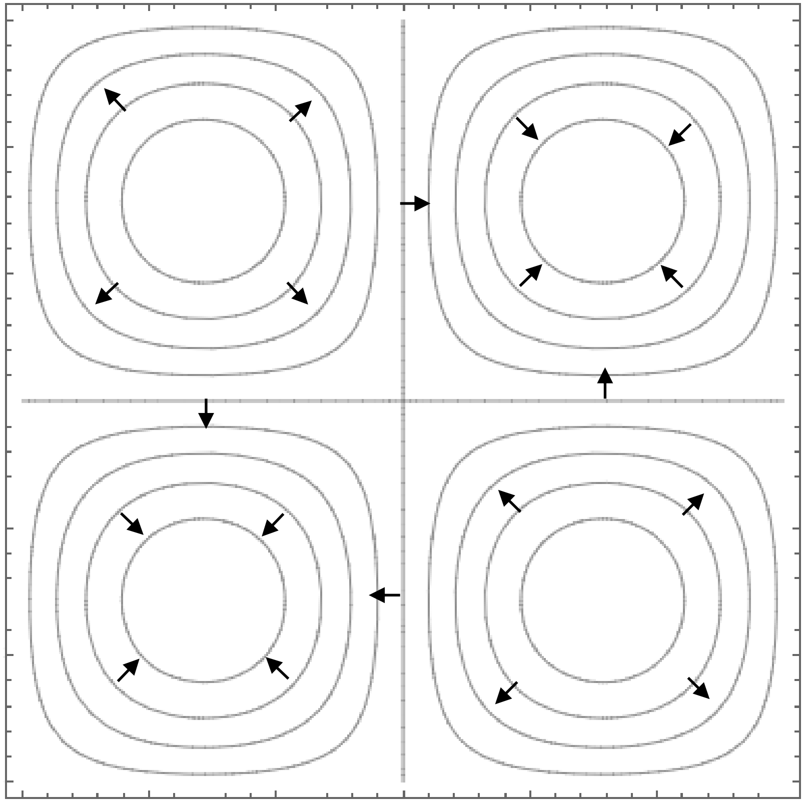




$$\int_a^b f(x)\,dx$$

Integration

df



Covector Field/
Differential Form

Differential Form interpretation of Integrals

Every (single) integral….

$$\int_a^b f(x)\,dx$$

$$\iint dx\,dy$$

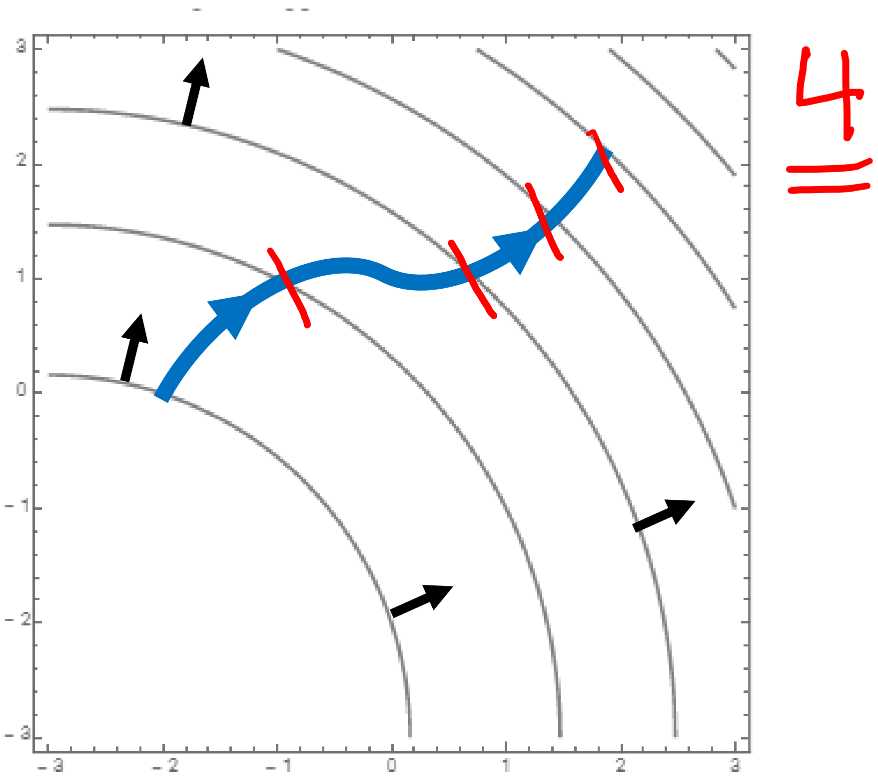
$$\iiint dx\,dy\,dz$$

Differential Form interpretation of Integrals

Every (single) integral involves

- a differential form (covector field)
- a path

$$\int_a^b f(x)dx$$



The result of the integral is just the number of covector stacks pierced by the path.

Fundamental Theorem of Calculus

$$\int_a^b f(x)dx = F(b) - F(a)$$

F is the anti-derivative of f

$$\int_a^b \frac{dF}{dx} dx = F(b) - F(a)$$

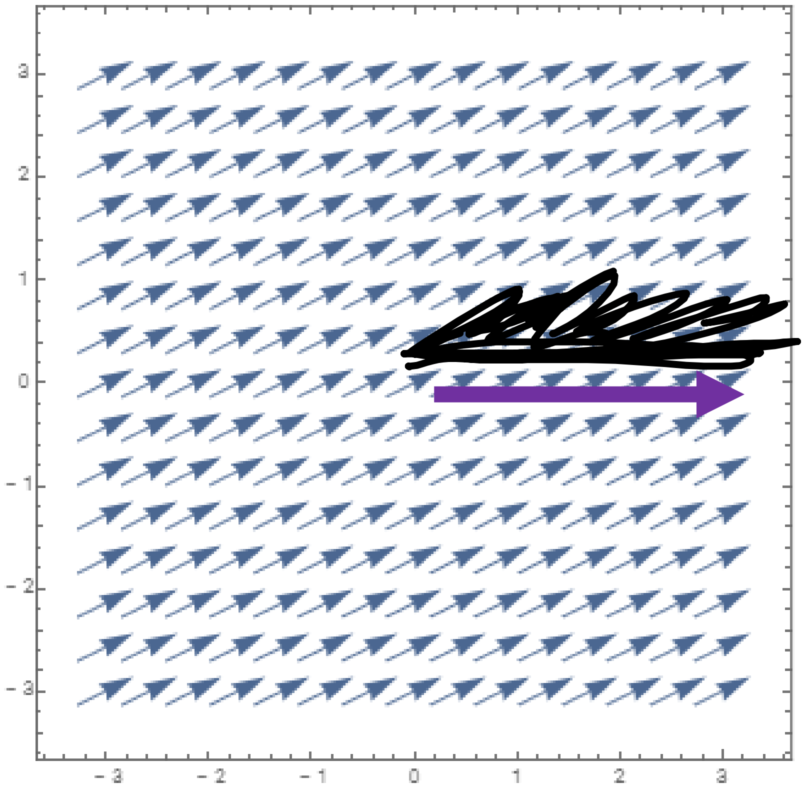
Fundamental Theorem of Calculus for Line Integrals ("Gradient Theorem")

$$\int_{P[a,b]} \nabla F \cdot d\vec{r} = F(b) - F(a)$$

Work

Force Field

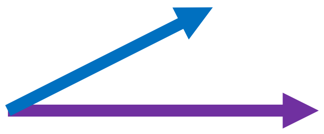
$$\vec{F} = 2\overrightarrow{e_x} + 1\overrightarrow{e_y}$$



Work done by Force Field

$$W = \vec{F} \cdot \vec{R}$$

$$\vec{R} = 3\overrightarrow{e_x} + 0\overrightarrow{e_y}$$



$$W = \vec{F} \cdot \vec{R}$$

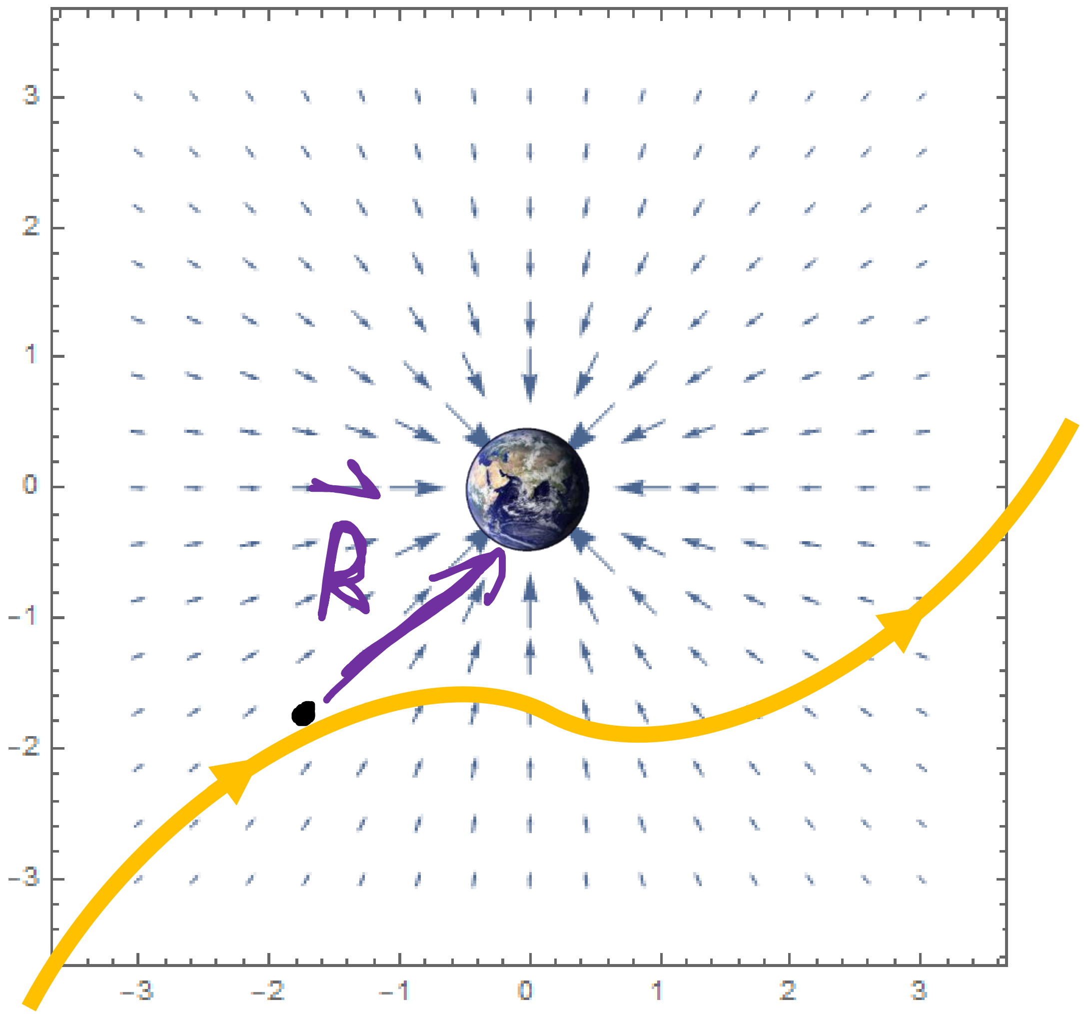
$$W = \left(2\overrightarrow{e_x} + 1\overrightarrow{e_y}\right) \cdot \left(3\overrightarrow{e_x} + 0\overrightarrow{e_y}\right)$$

$$W = (2)(3) + (1)(0)$$

$$W = 6$$ Joules

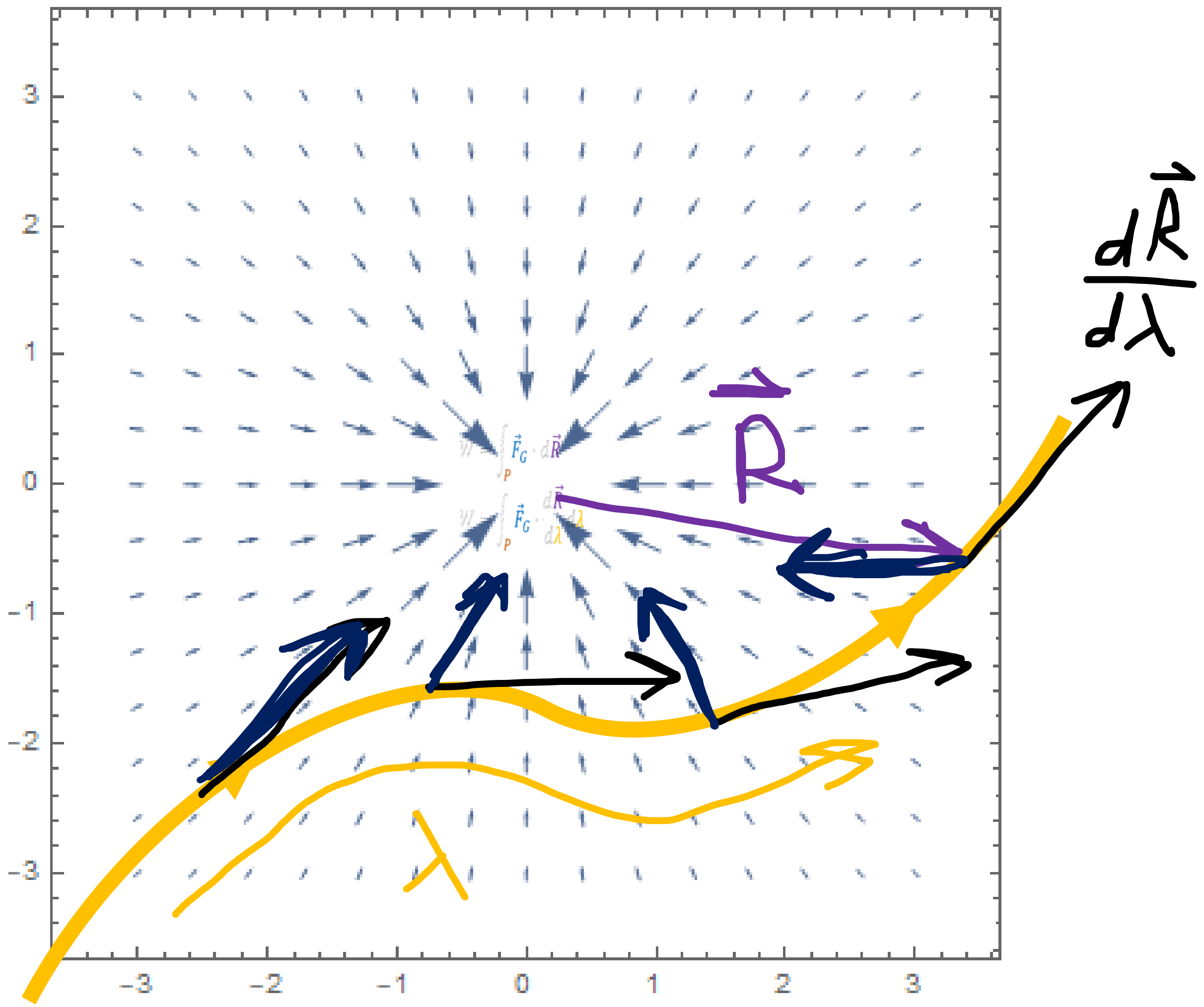
Gravitational Force Field

$$\vec{F}_G = \frac{GMm}{\|\vec{R}\|^2}\left(-\overrightarrow{e_r}\right)$$



Gravitational Force Field

$$\vec{F}_G = \frac{GMm}{\|\vec{R}\|^2}(-\vec{e_r})$$



Work done by Force Field

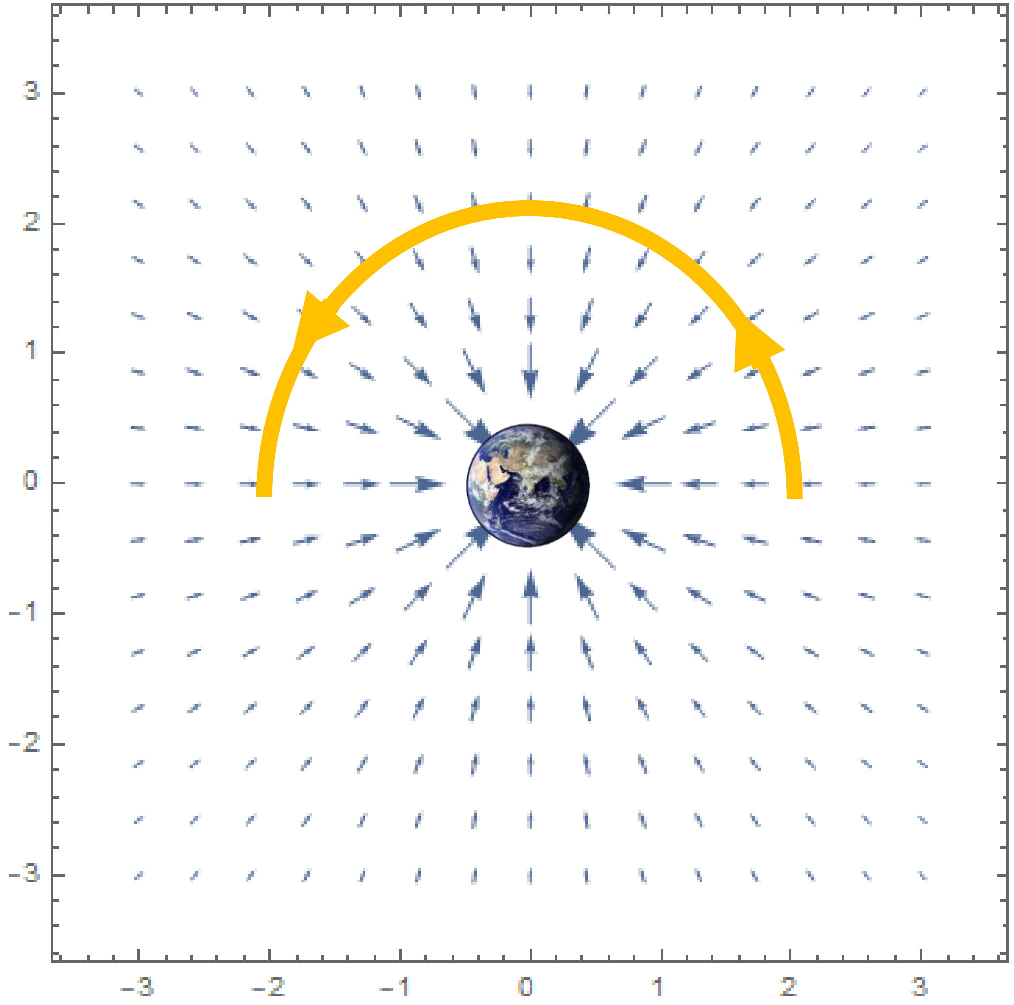
~~$W = \vec{F} \cdot \vec{R}$~~

$$W = \int_P \vec{F}_G \cdot d\vec{R}$$

$$W = \int_P \vec{F}_G \cdot \frac{d\vec{R}}{d\lambda} d\lambda$$

Gravitational Force Field

$$\vec{F}_G = \frac{GMm}{\|\vec{R}\|^2}(-\vec{e_r})$$



$$\vec{R}(\lambda) = (r = 2, \theta = \lambda)$$

$$\frac{d\vec{R}}{d\lambda} = \frac{dr}{d\lambda}\underbrace{\frac{d\vec{R}}{dr}}_{\vec{e_r}} + \frac{d\theta}{d\lambda}\underbrace{\frac{d\vec{R}}{d\theta}}_{\vec{e_\theta}}$$

$$\frac{d\vec{R}}{d\lambda} = 0\vec{e_r} + 1\vec{e_\theta}$$

$$W = \int_P \vec{F}_G \cdot \frac{d\vec{R}}{d\lambda} d\lambda$$

$$W = \int_P \left(\frac{GMm}{\|\vec{R}\|^2}(-\vec{e_r})\right) \cdot (\vec{e_\theta})\, d\lambda$$

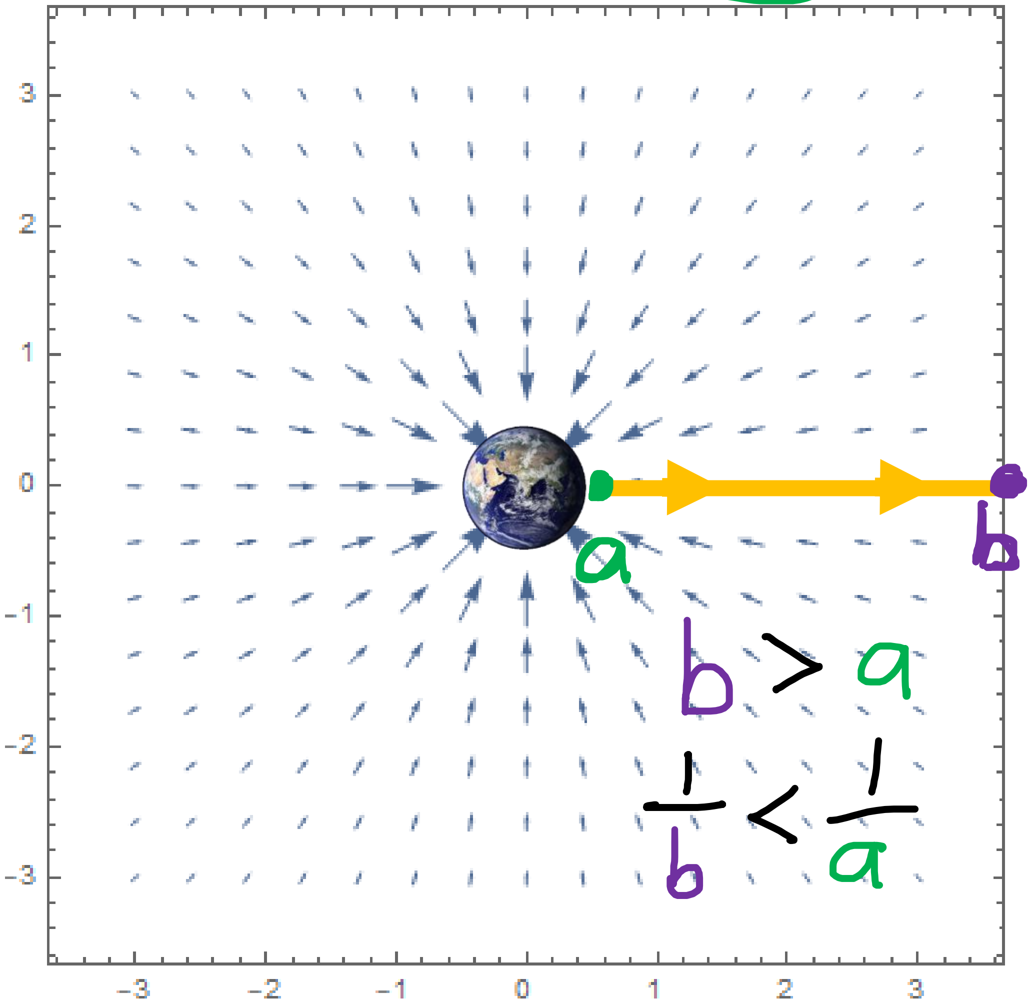
$$W = -\frac{GMm}{4}\int_C (\vec{e_r} \cdot \vec{e_\theta})\, d\lambda = 0$$

Gravitational Force Field

$$\vec{F}_G = \frac{GMm}{\|\vec{R}\|^2}(-\overrightarrow{e_r})$$

$$\vec{R}(\lambda) = (r = \lambda, \theta = 0)$$

$$\frac{d\vec{R}}{d\lambda} = 1\overrightarrow{e_r} + 0\overrightarrow{e_\theta}$$



$$W = \int_P \vec{F}_G \cdot \frac{d\vec{R}}{d\lambda} d\lambda$$

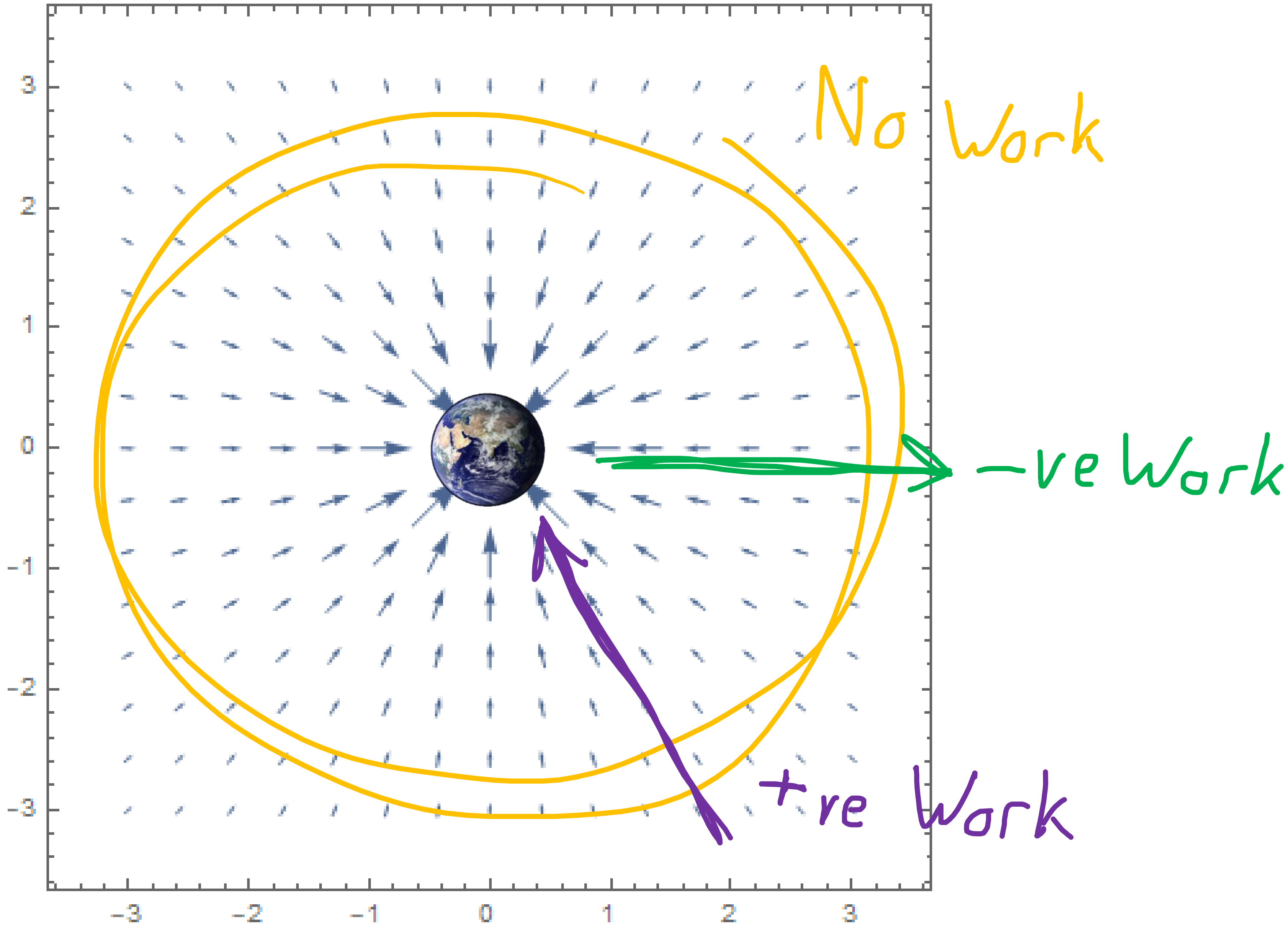
$$W = \int_P \left(\frac{GMm}{\|\vec{R}\|^2}(-\overrightarrow{e_r})\right) \cdot (\overrightarrow{e_r})\, d\lambda$$

$$W = GMm \int_P \left(-\frac{1}{\lambda^2}\right)(\overrightarrow{e_r} \cdot \overrightarrow{e_r})\, d\lambda$$

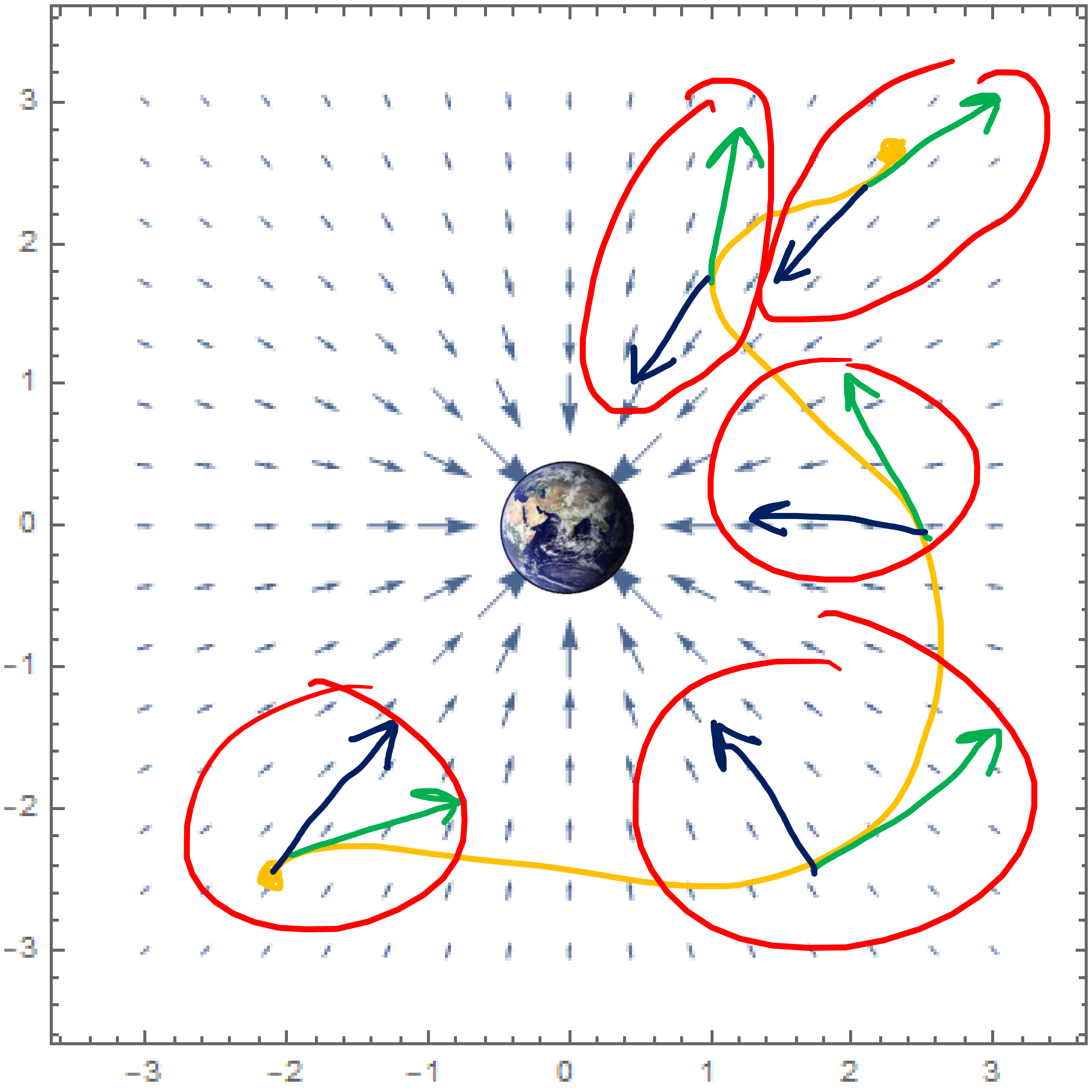
1

$$W = GMm\left[\frac{1}{\lambda}\right]_a^b = GMm\left(\frac{1}{b} - \frac{1}{a}\right)$$

(Negative Result)


No Work
-ve Work
+ve Work
3
2
1
0
-1
-2
-3
-3
-2
-1
0
1
2
3

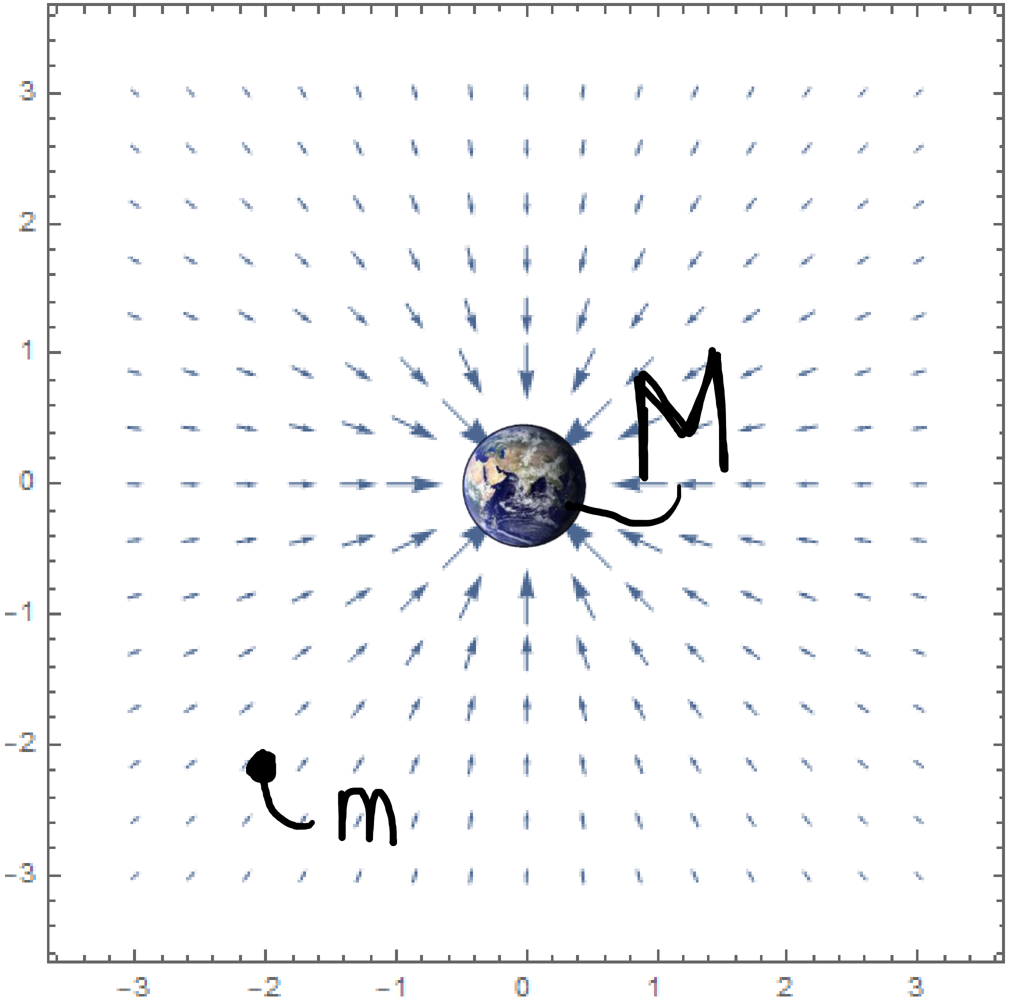
$$\int \vec{F}_G \cdot \frac{d\vec{R}}{d\lambda} \, d\lambda$$



Gravitational Potential

Gravitational Force Field

$$\vec{F}_G = \frac{GMm}{\|\vec{R}\|^2}(-\vec{e_r})$$



$$\vec{F}_G = m\frac{GM}{\|\vec{R}\|^2}(-\vec{e_r})$$

$$\vec{F}_G = m\vec{G}$$

Gravitational Field

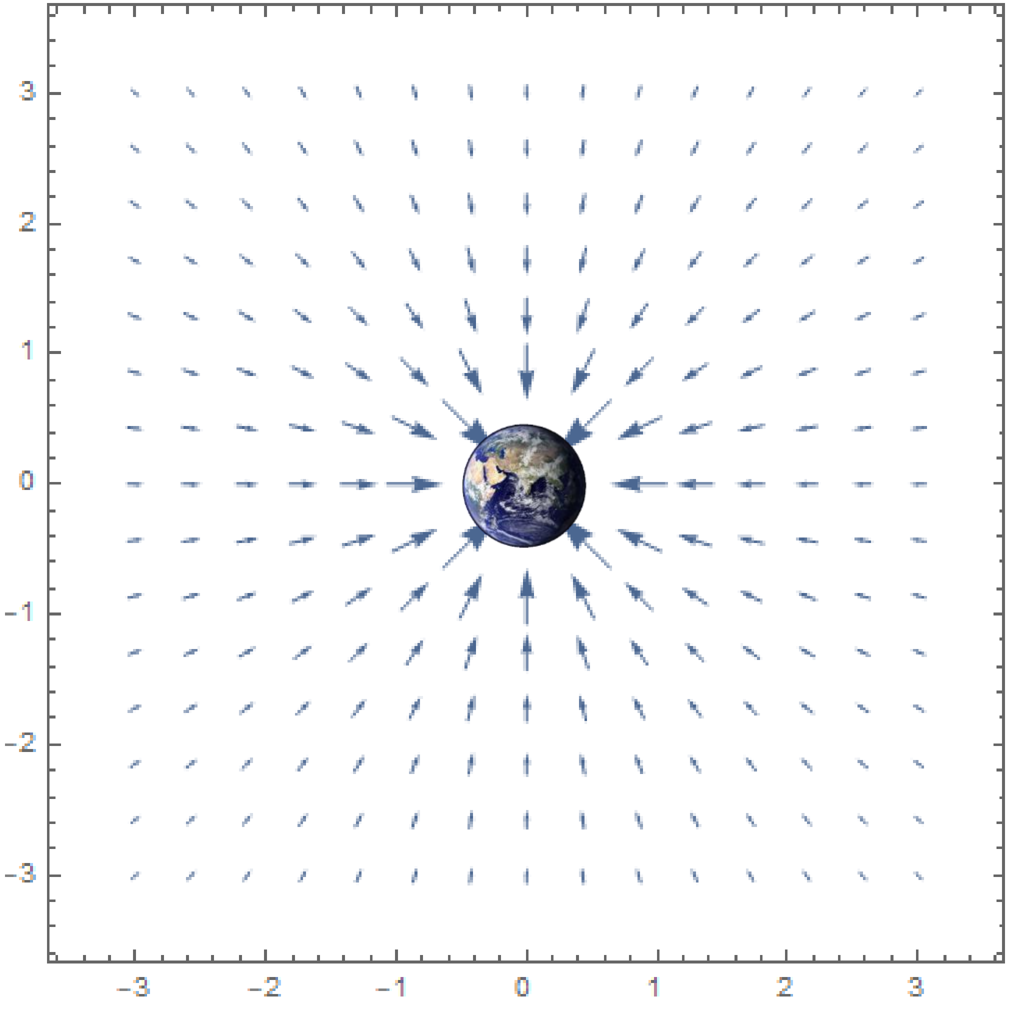
$$\vec{G} = -\nabla\phi$$

Gravitational Potential

$$\vec{F}_G = m(-\nabla\phi)$$

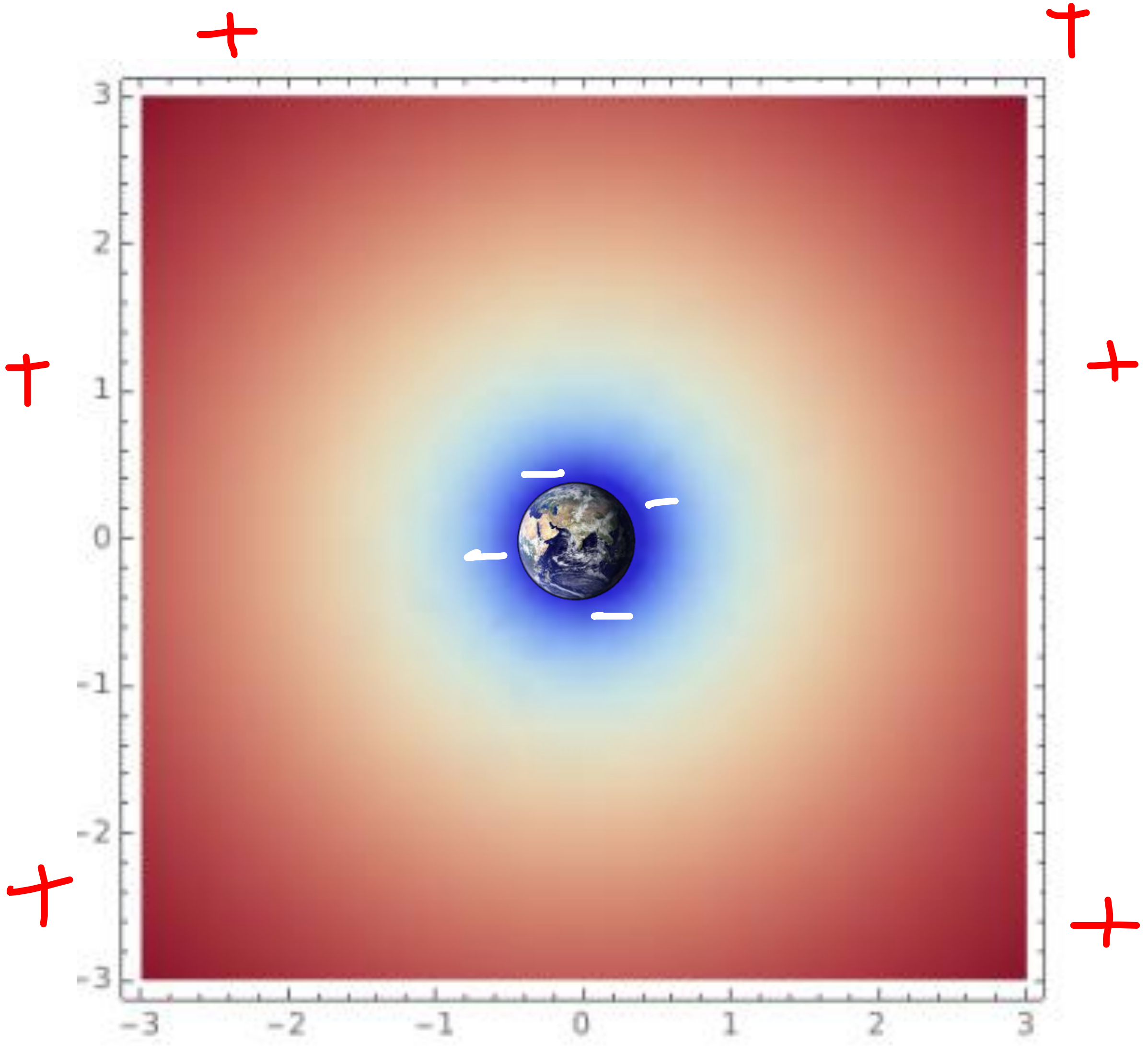
Gravitational Force Field

$$\vec{F}_G = m(-\nabla\phi)$$



Gravitational Potential

$$\phi$$

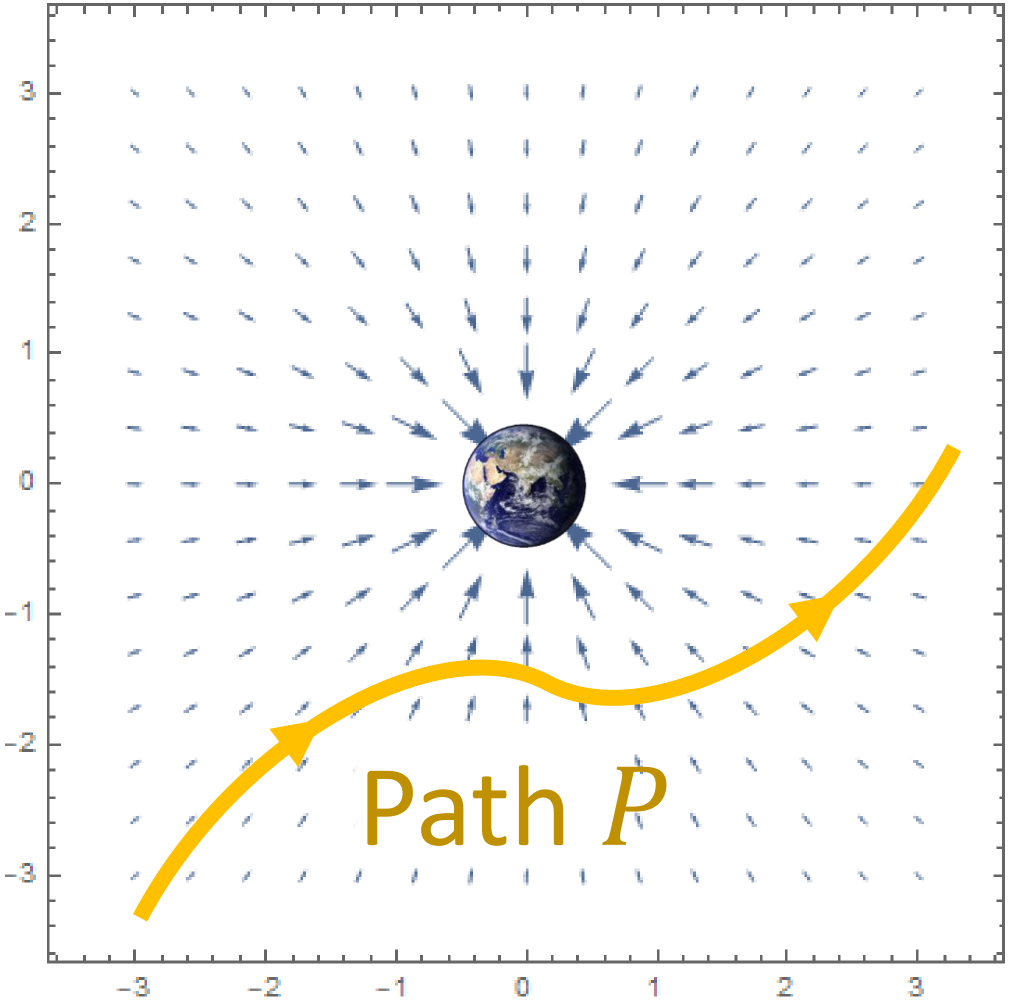


Gravitational Force Field

$$\vec{F}_G = m(-\nabla\phi)$$

$$W = \int_P \vec{F}_G \cdot \frac{d\vec{R}}{d\lambda} d\lambda$$

$$W = \int_P m(-\nabla\phi) \cdot \frac{d\vec{R}}{d\lambda} d\lambda$$



$$\nabla\phi = \frac{\partial\phi}{\partial x}\overrightarrow{e_x} + \frac{\partial\phi}{\partial y}\overrightarrow{e_y}$$

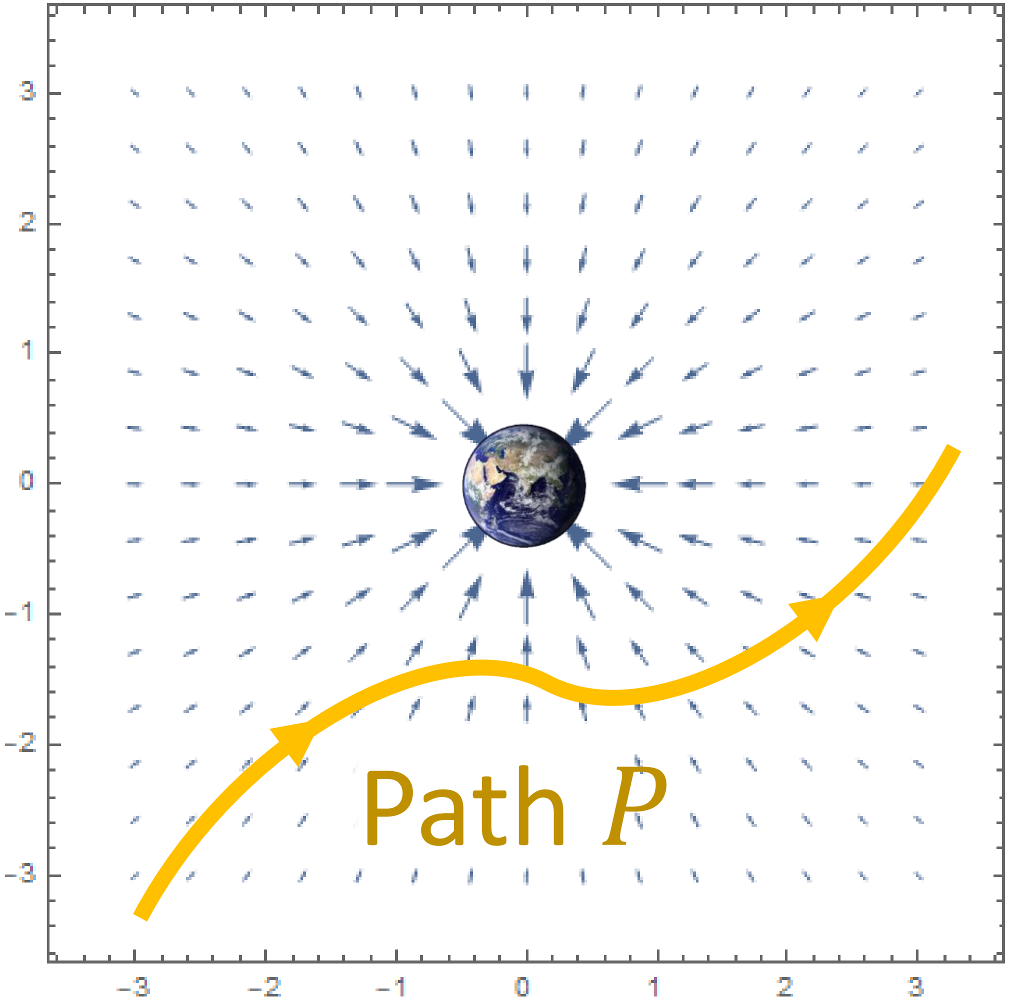
Cartesian Only!

$$\frac{d\vec{R}}{d\lambda} = \frac{dx}{d\lambda}\underbrace{\frac{d\vec{R}}{dx}}_{\vec{e}_x} + \frac{dy}{d\lambda}\underbrace{\frac{d\vec{R}}{dy}}_{\vec{e}_y}$$

$$\frac{d\vec{R}}{d\lambda} = \frac{dx}{d\lambda}\overrightarrow{e_x} + \frac{dy}{d\lambda}\overrightarrow{e_y}$$

Gravitational Force Field

$$\vec{F}_G = m(-\nabla\phi)$$



$$W = \int_P \vec{F}_G \cdot \frac{d\vec{R}}{d\lambda} d\lambda$$

$$W = \int_P m(-\nabla\phi) \cdot \frac{d\vec{R}}{d\lambda} d\lambda$$

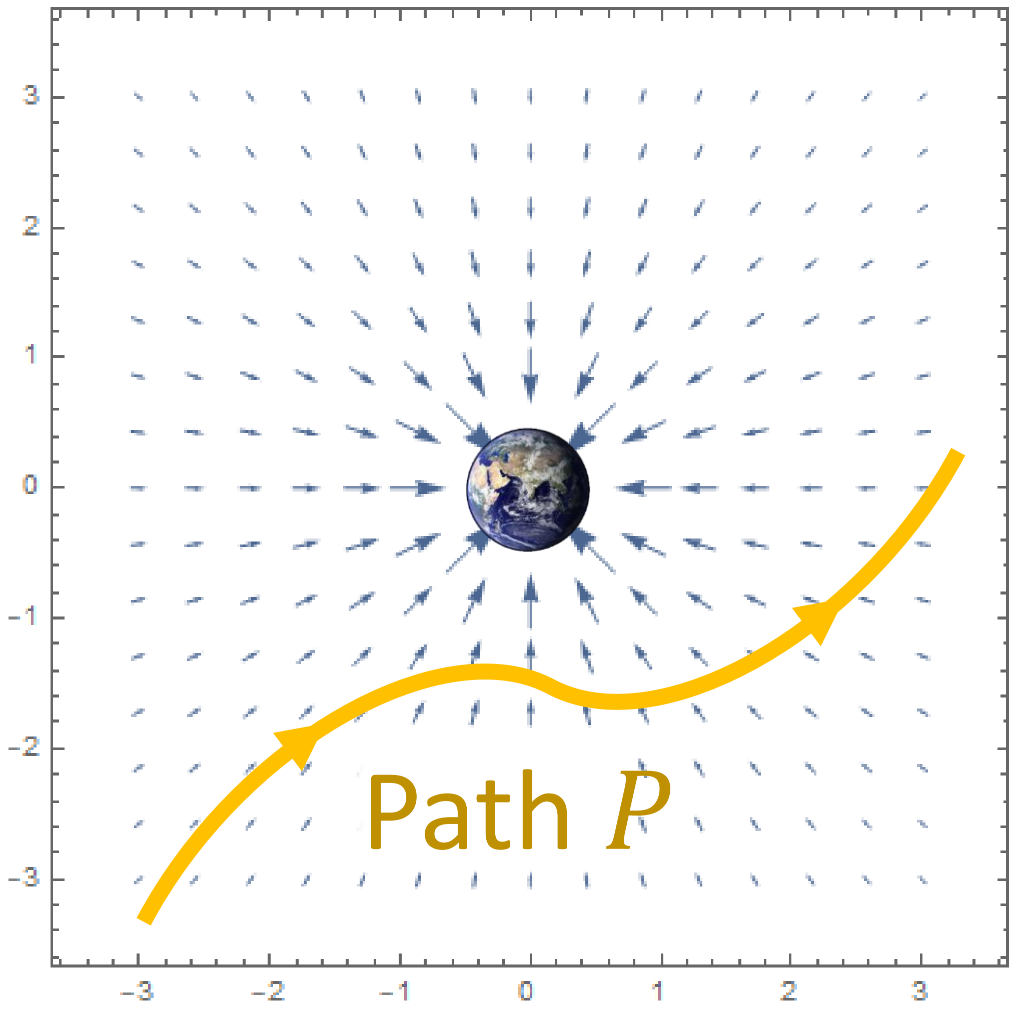
$$\nabla\phi \cdot \frac{d\vec{R}}{d\lambda}$$

$$= \left(\frac{\partial\phi}{\partial x}\vec{e_x} + \frac{\partial\phi}{\partial y}\vec{e_y}\right) \cdot \left(\frac{dx}{d\lambda}\vec{e_x} + \frac{dy}{d\lambda}\vec{e_y}\right)$$

$$= \frac{\partial\phi}{\partial x}\frac{dx}{d\lambda} + \frac{\partial\phi}{\partial y}\frac{dy}{d\lambda} = \frac{d\phi}{d\lambda}$$

Gravitational Force Field

$$\vec{F}_G = m(-\nabla\phi)$$



$$W = \int_P \vec{F}_G \cdot \frac{d\vec{R}}{d\lambda} d\lambda$$

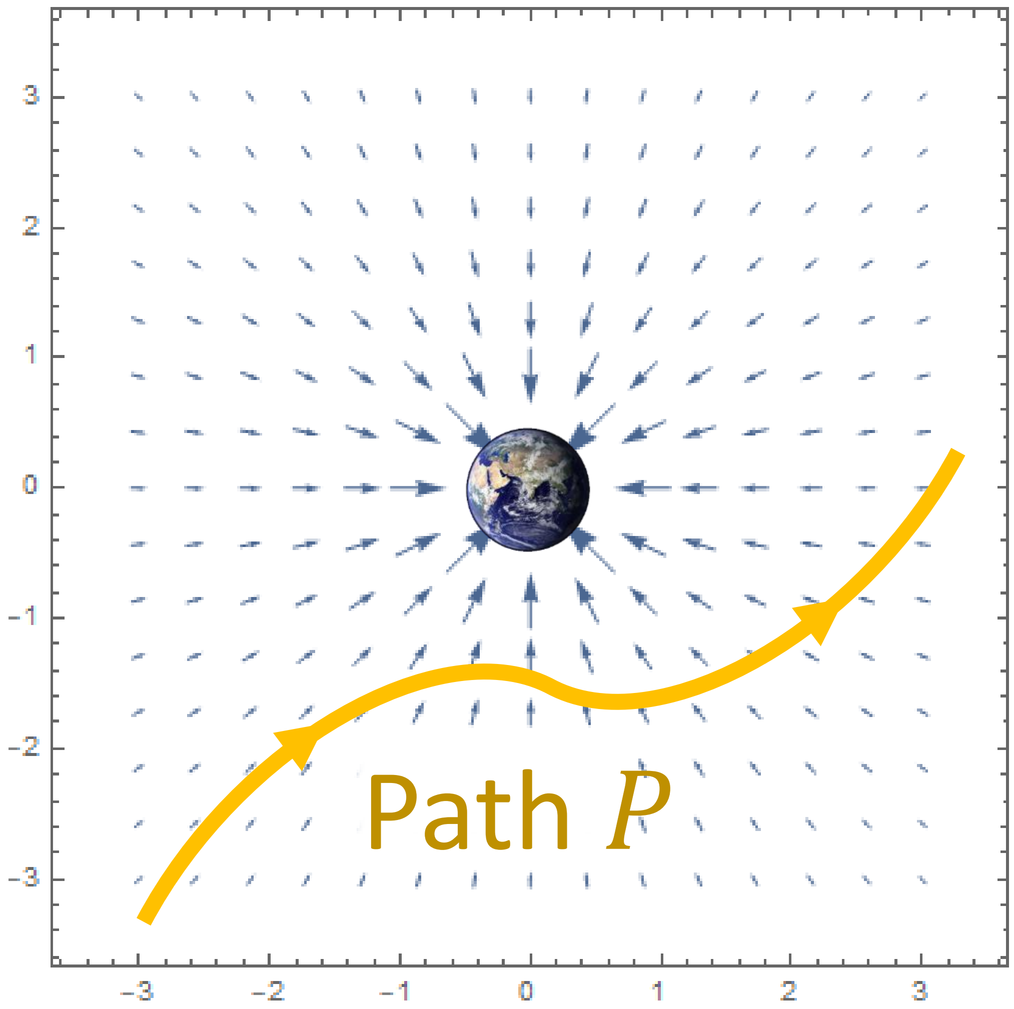
$$W = \int_P m(-\nabla\phi) \cdot \frac{d\vec{R}}{d\lambda} d\lambda$$

$$W = -m \int_P \frac{d\phi}{d\lambda} d\lambda$$

$$W = -m \int_P d\phi$$

Gravitational Force Field

$$\vec{F}_G = m(-\nabla\phi)$$



$$W = \int_C \vec{F}_G \cdot \frac{d\vec{R}}{d\lambda} d\lambda$$

$$W = \int_P m(-\nabla\phi) \cdot \frac{d\vec{R}}{d\lambda} d\lambda$$

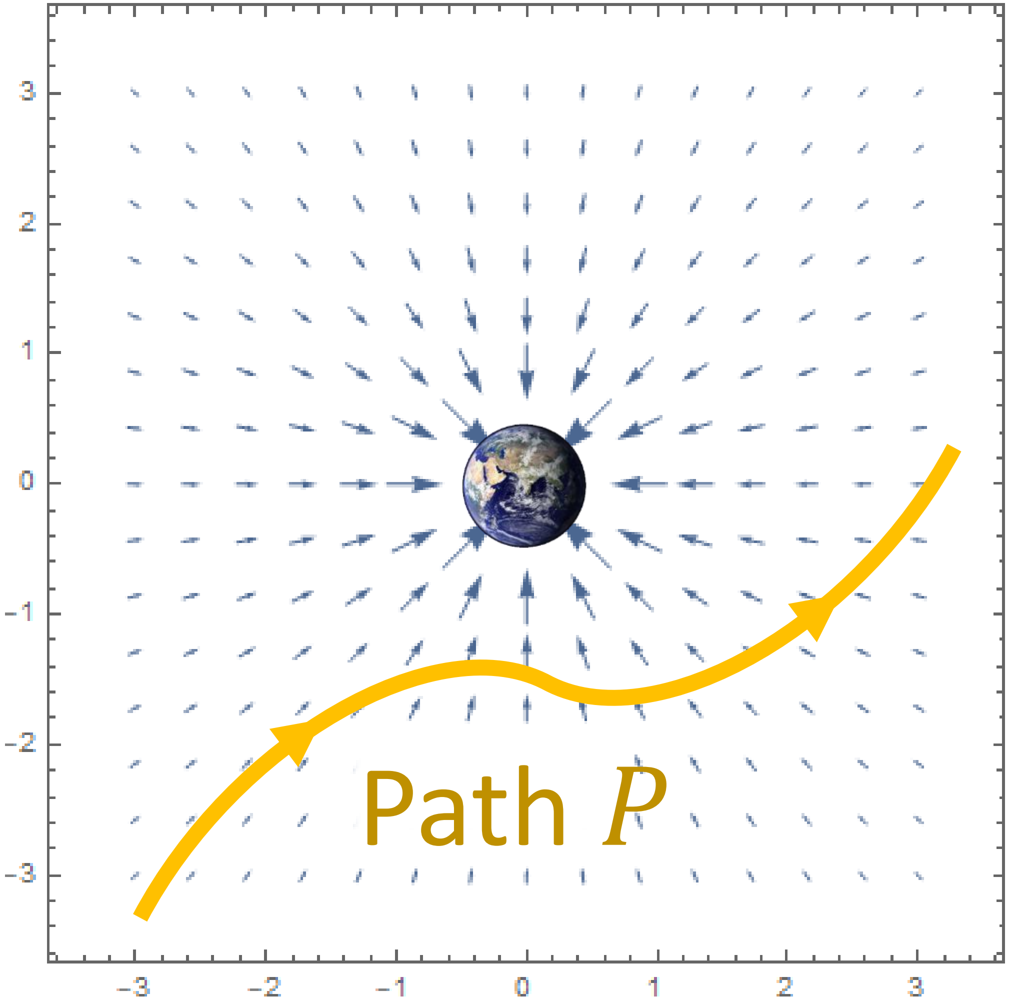
$$W = -m \int_P \frac{d\phi}{d\lambda} d\lambda$$

$$W = -m \int_P d\phi$$

$$W = -m[\phi(P_{end}) - \phi(P_{start})]$$

Gravitational Force Field

$$\vec{F}_G = m(-\nabla\phi)$$



$$W = \int_C \vec{F}_G \cdot \frac{d\vec{R}}{d\lambda} d\lambda$$

$$W = \int_P m(-\nabla\phi) \cdot \frac{d\vec{R}}{d\lambda} d\lambda$$

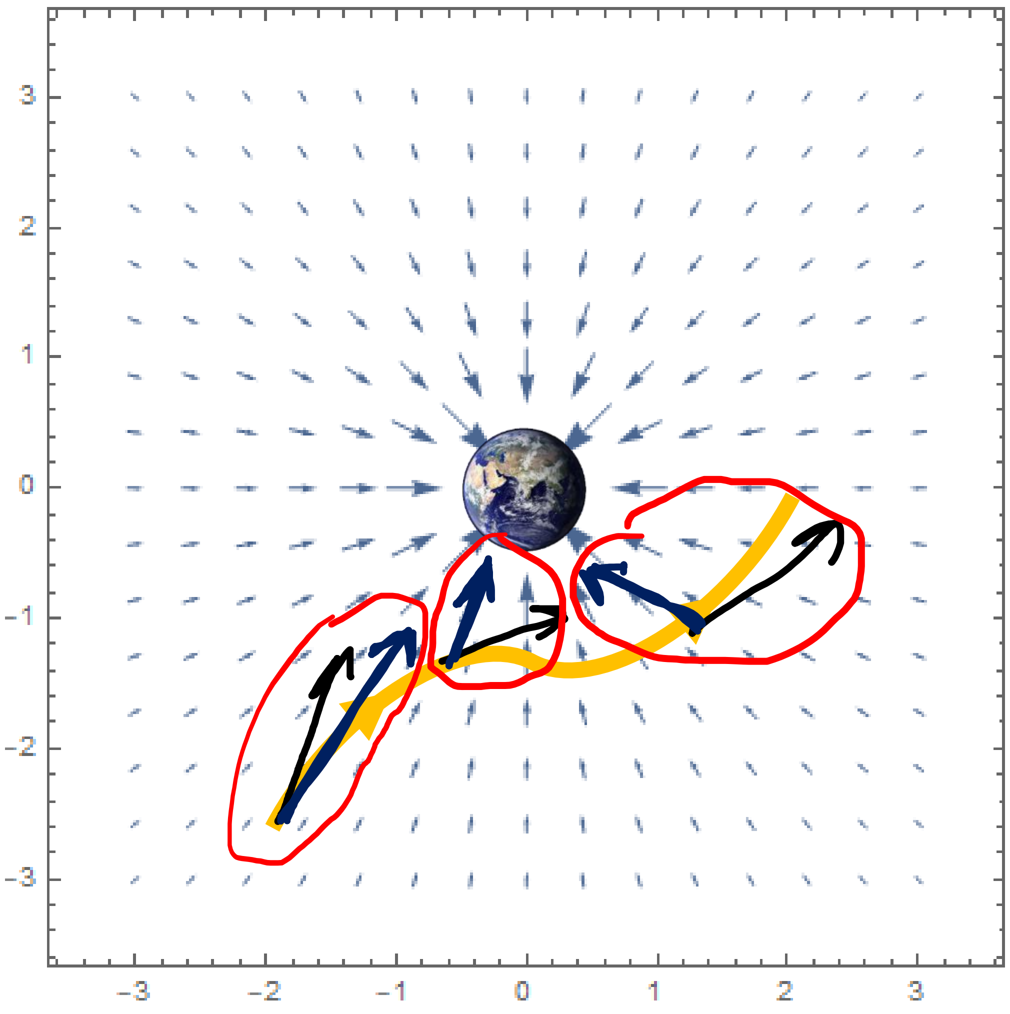
$$W = -m \int_P \frac{d\phi}{d\lambda} d\lambda$$

$$W = -m \int_P d\phi$$

$$W = -m[\phi(P_{end}) - \phi(P_{start})]$$

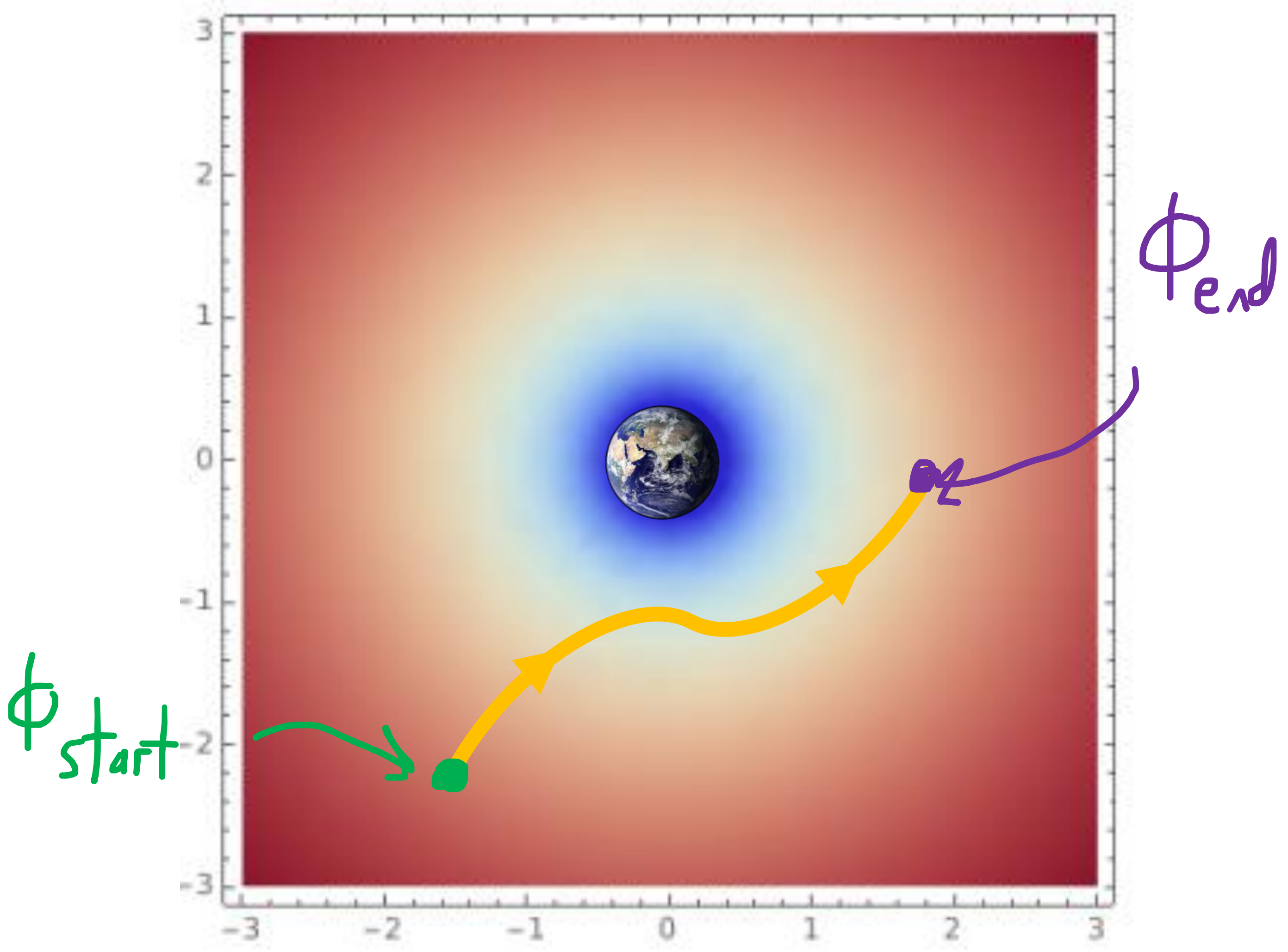
Gravitational Force Field

$$\vec{F}_G = m(-\nabla\phi)$$



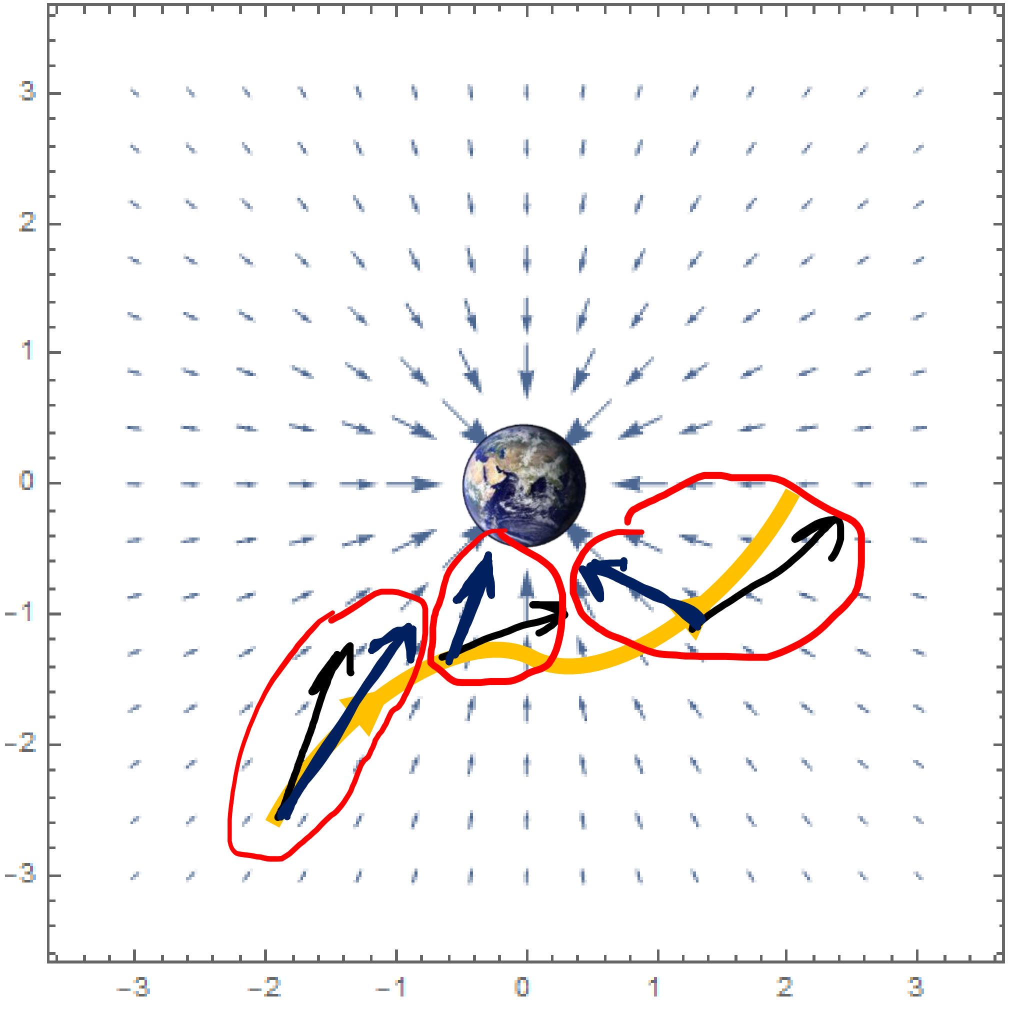
Gravitational Potential

$$\phi$$



Gravitational Force Field

$$\vec{F}_G = m(-\nabla\phi)$$

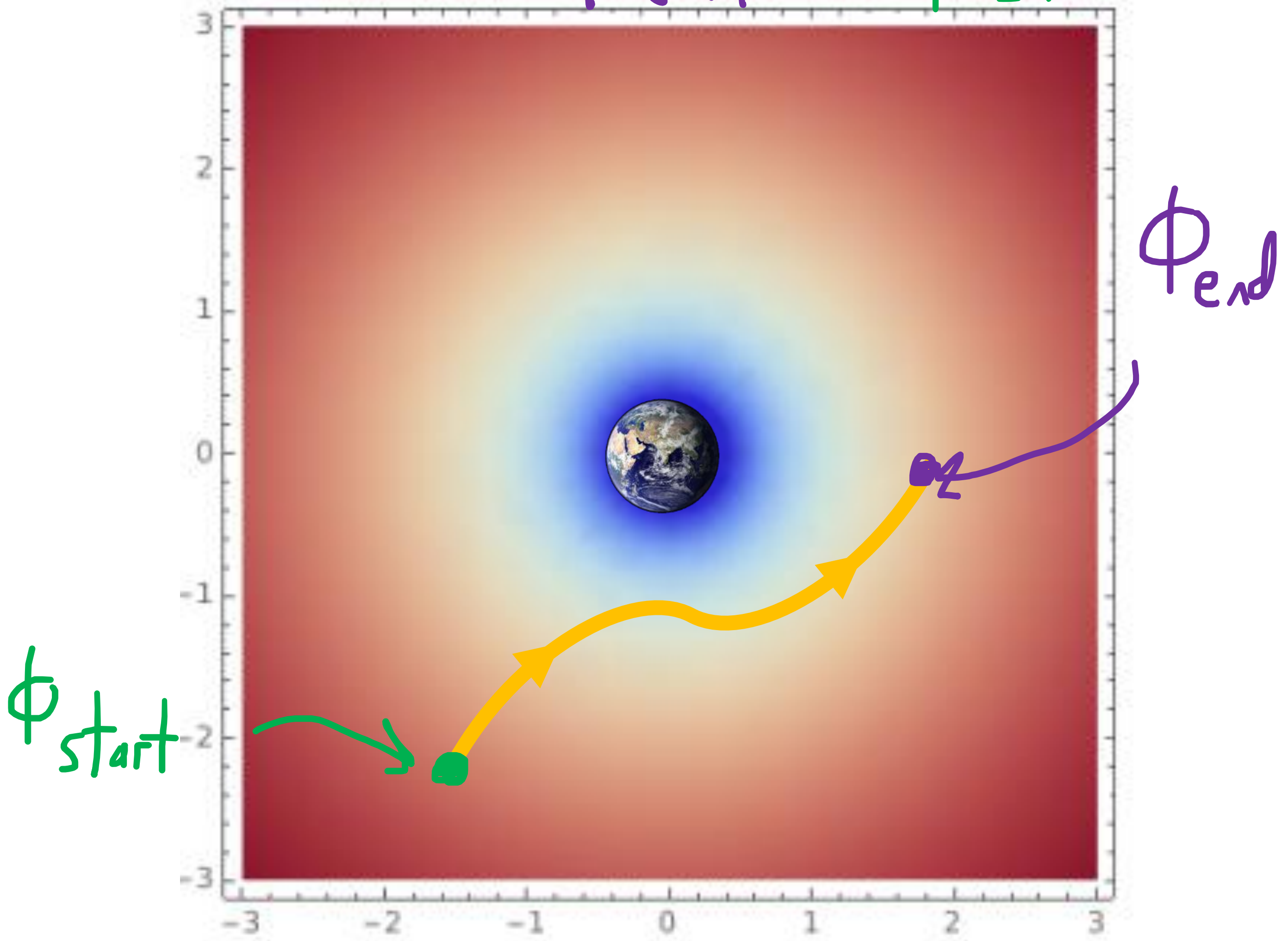


Gravitational Potential

ϕ

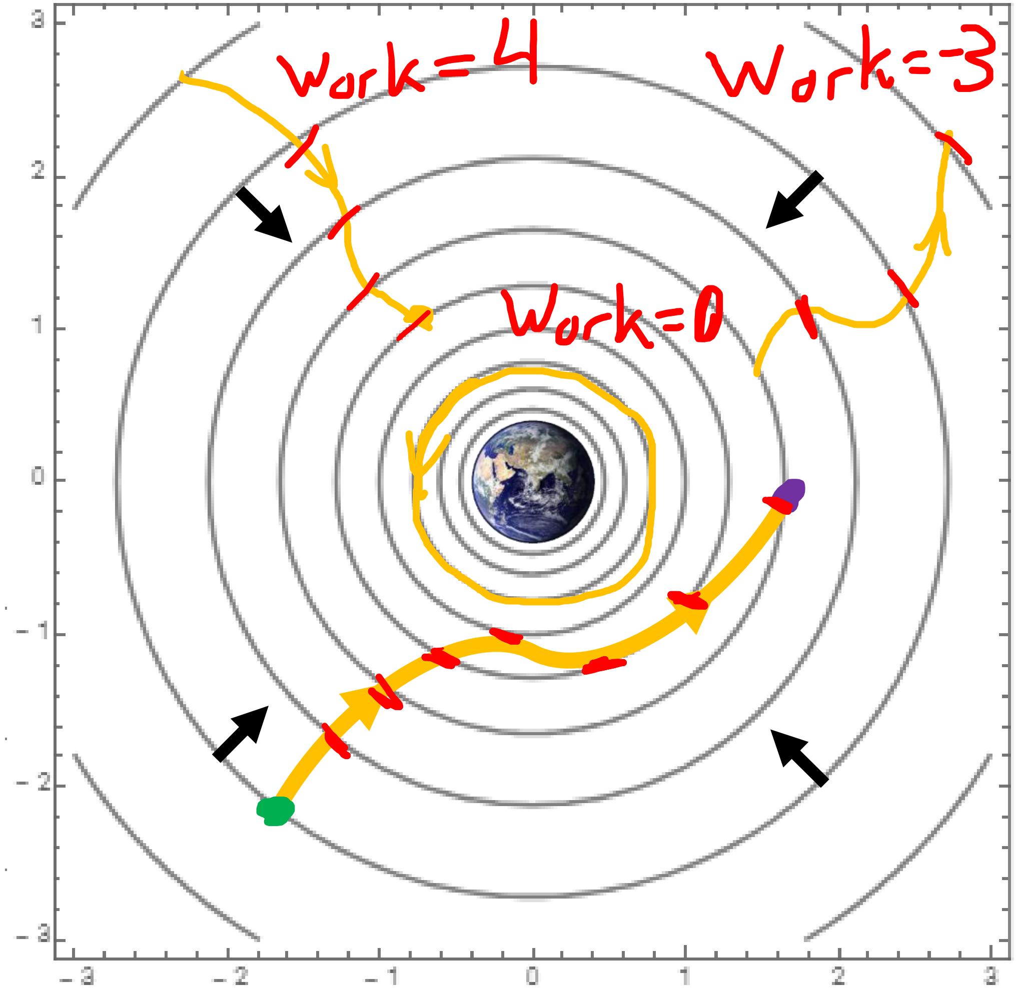
(if m=1)

$\text{Work} = \phi_{end} - \phi_{start}$



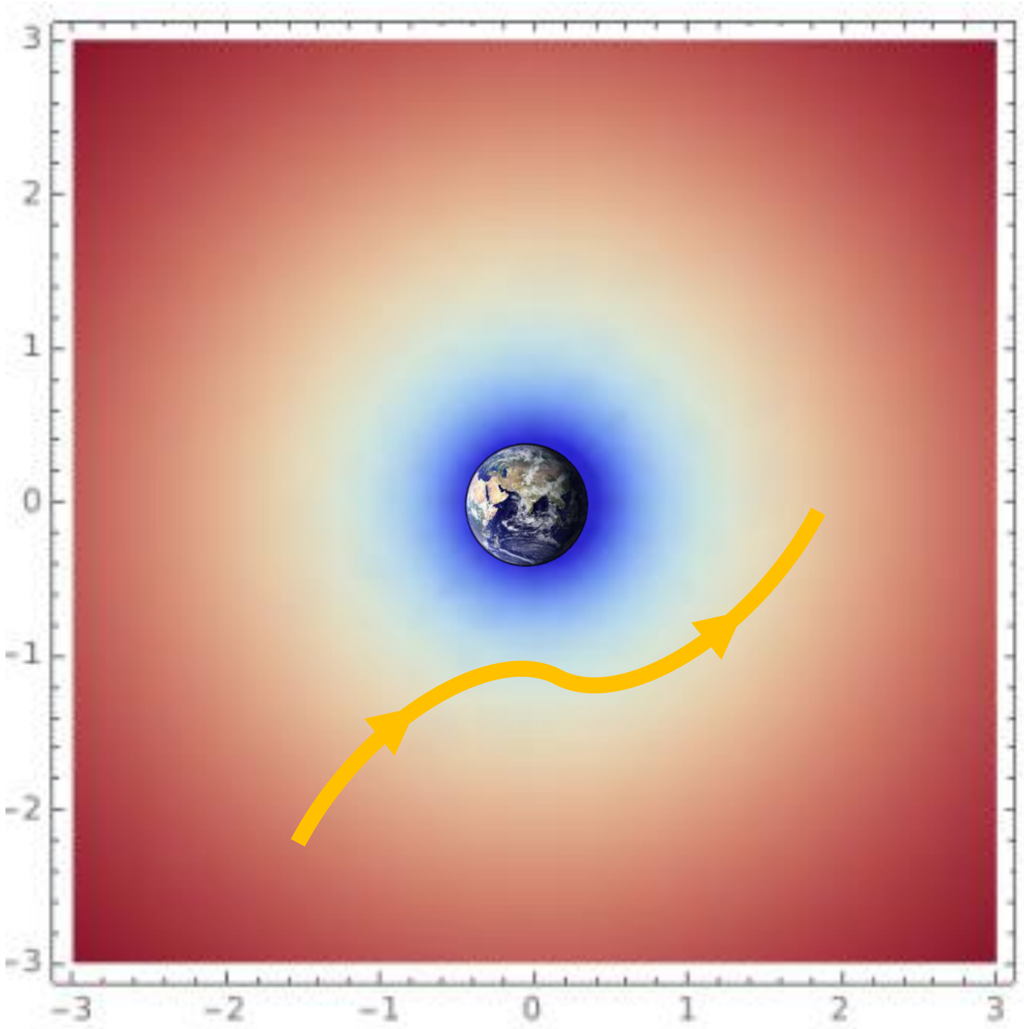
Gravitational Force (Covector) Field

$$dF_G = m(-d\phi)$$



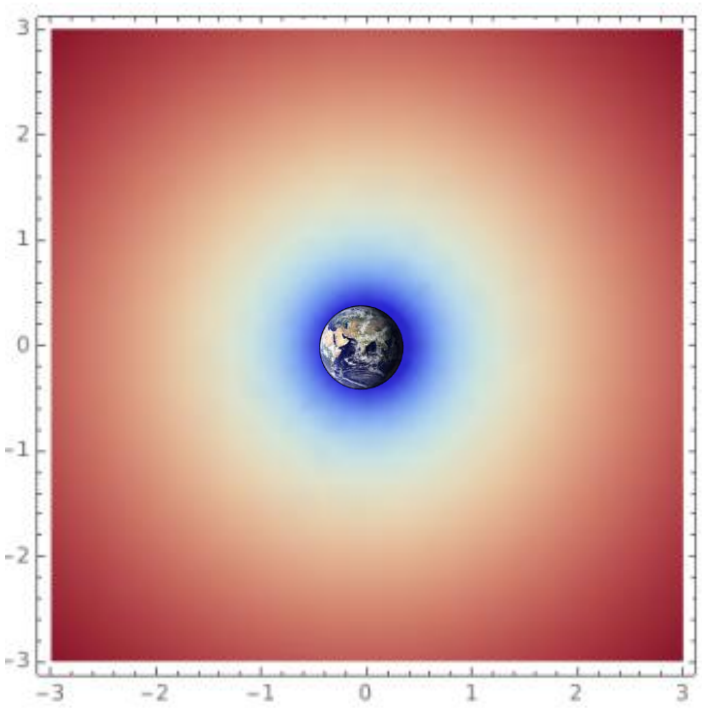
Gravitational Potential

$$\phi$$

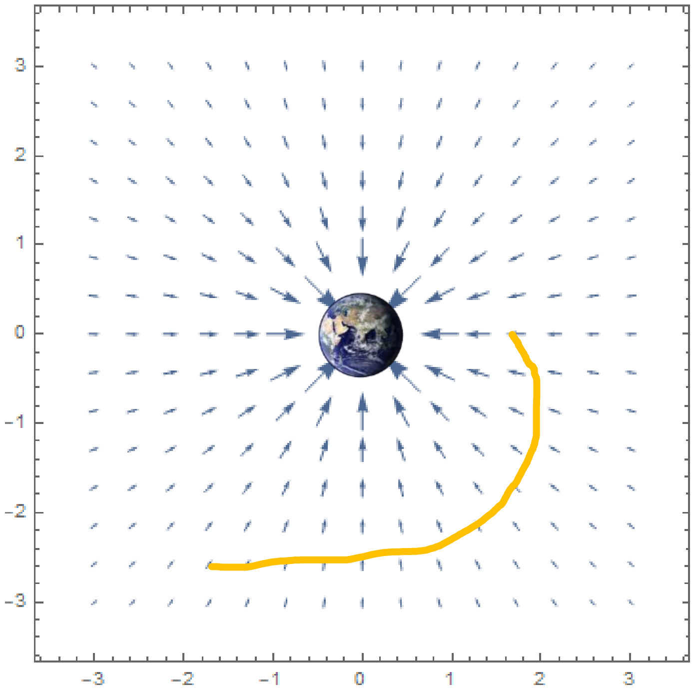


Gravitational Potential

ϕ



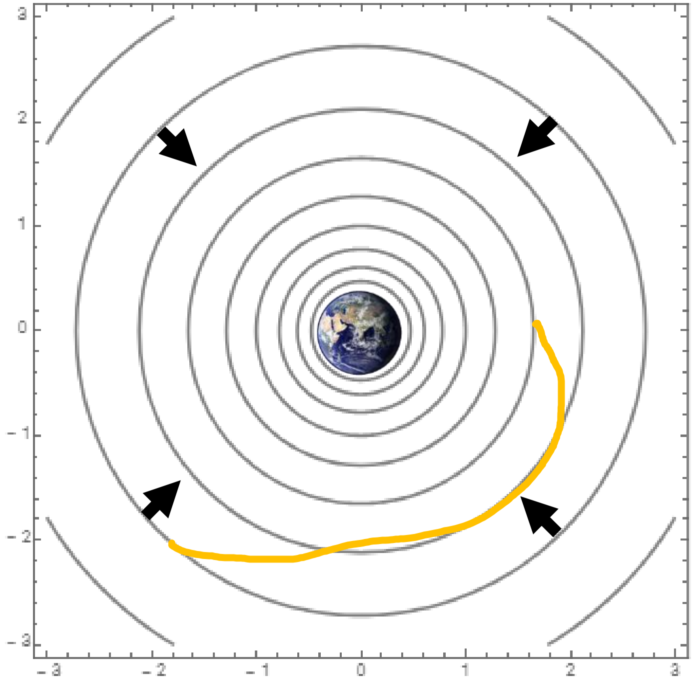
$-\nabla\phi$



Gravitational Force Vector Field

$\vec{F_G} = m(-\nabla\phi)$

$-d\phi$



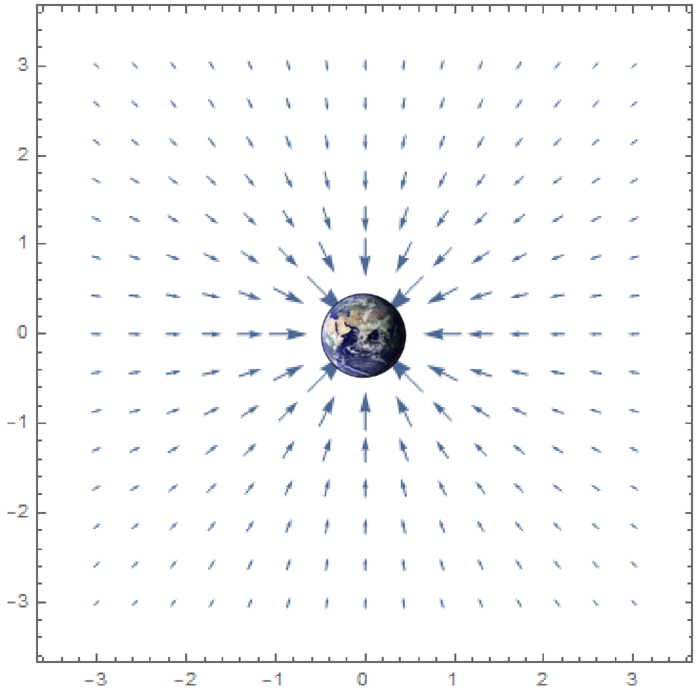
Gravitational Force Covector Field

$dF_G = m(-d\phi)$

Fundamental Theorem of Calculus for Line Integrals ("Gradient Theorem")

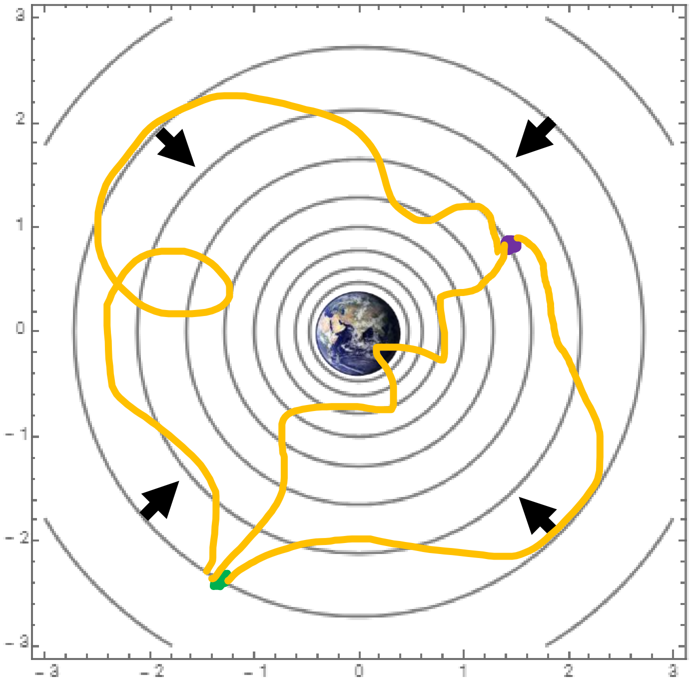
Old way (hard)

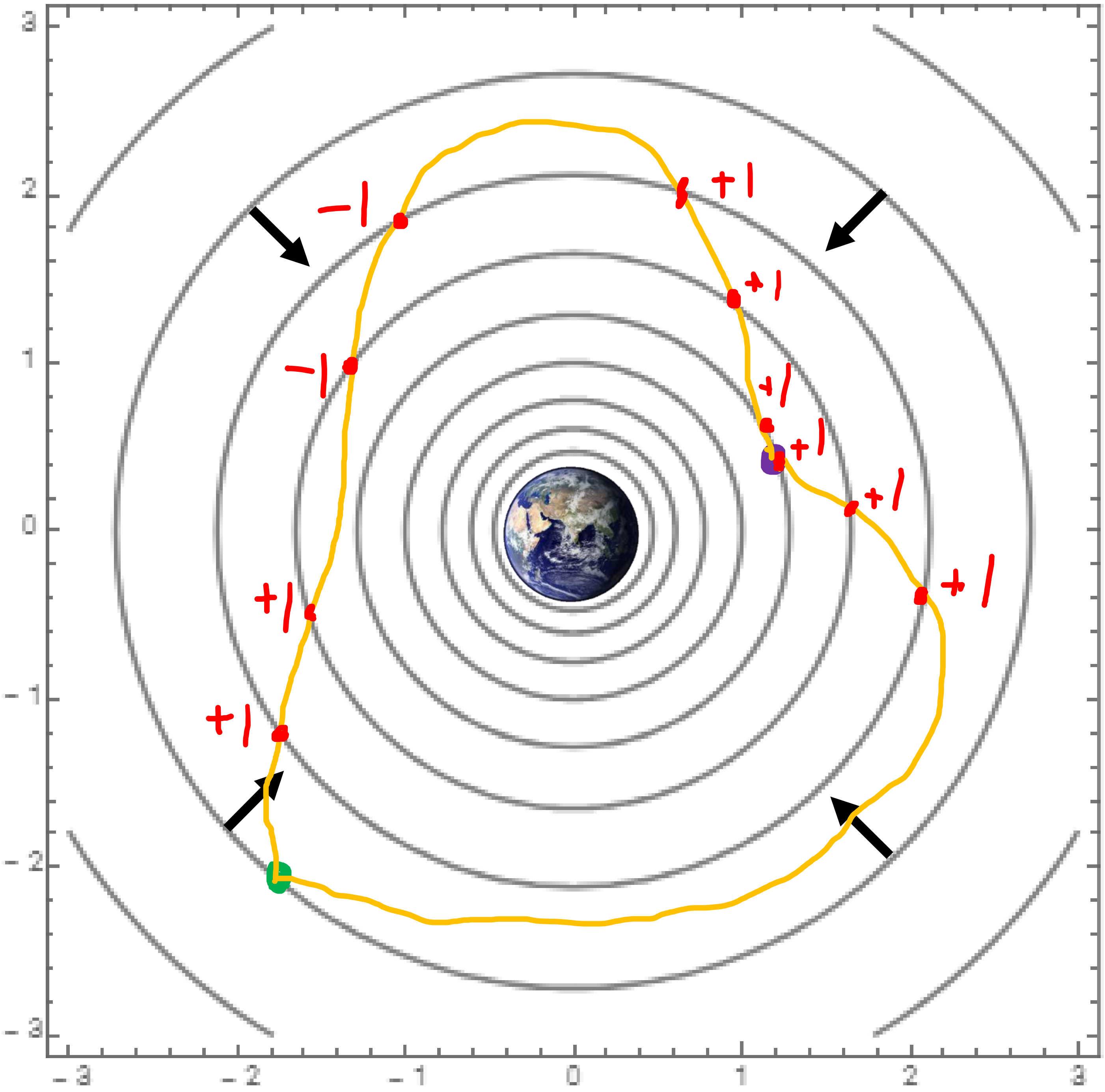
$$\int_{P[a,b]} \nabla\phi \cdot d\vec{R} = \phi(b) - \phi(a)$$



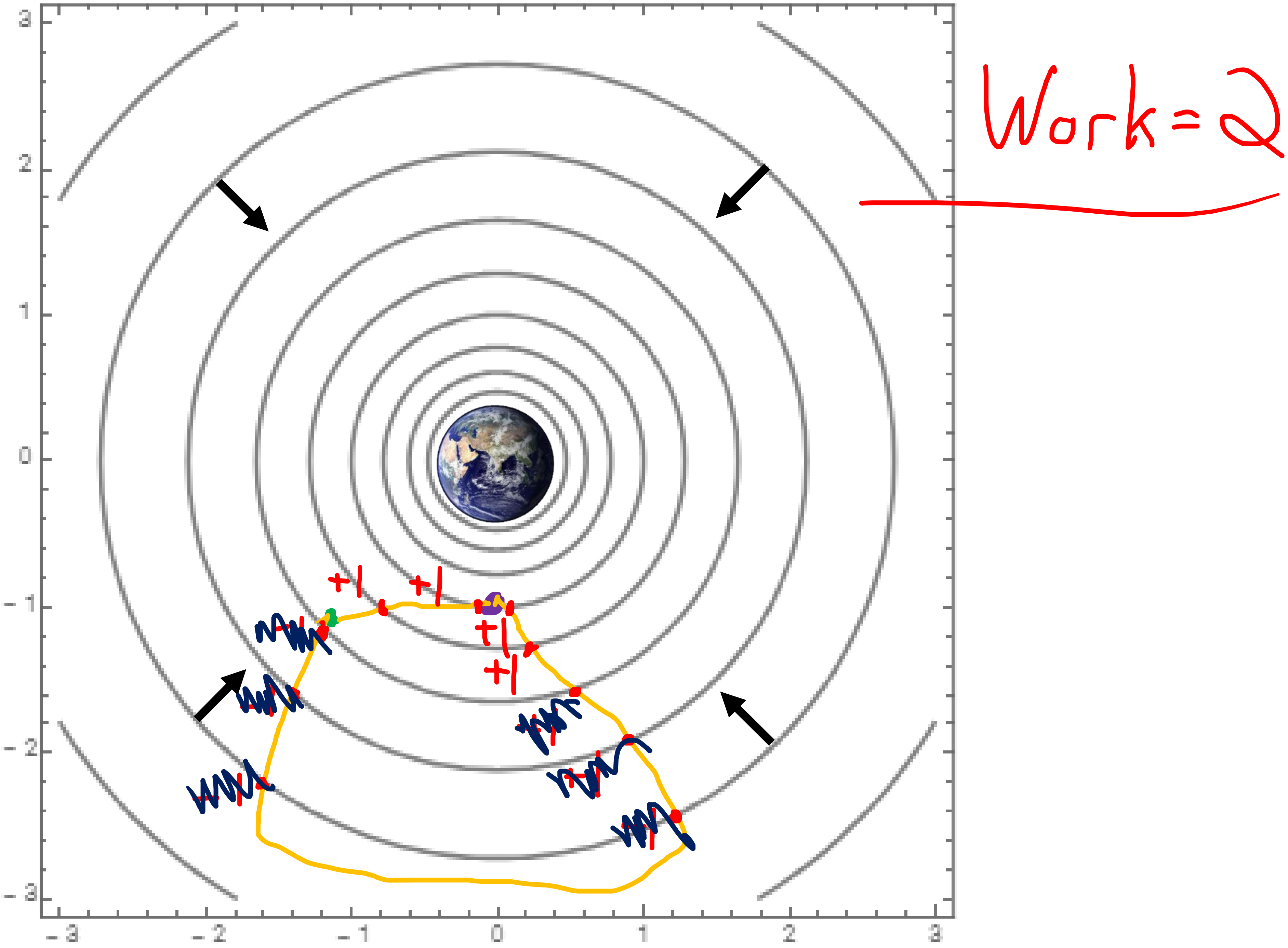
New way (easy)

$$\int_{P[a,b]} d\phi = \phi(b) - \phi(a)$$





3
2
1
0
−1
−2
−3
−3
−2
−1
0
1
2
3
−1
−1
+1
+1
+1
+1
+1
+1
+1
+1


Work=2
+1
+1
+1
+1

Differential Form interpretation of Integrals

Every (single) integral involves

- a path
- a covector field (differential form)

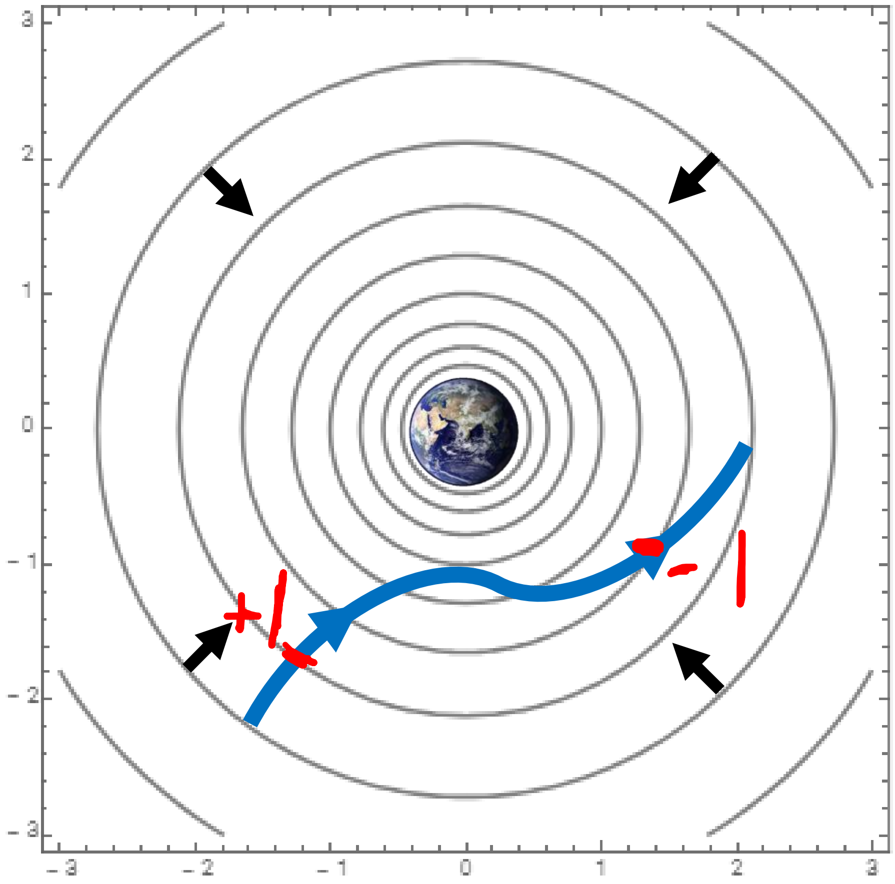
$$\int_{P[a,b]} y\,dx + x\,dy \qquad \int_0^2 (-6x+4)\,dx$$

Differential Form interpretation of Integrals

Every (single) integral involves

- a path
- a covector field (differential form)

$$\int_{P[a,b]} d\phi$$



The result of the integral is just the number of covector stacks pierced* by the path.

*(In the aligned direction.)