RStudio TinyTex install missing LaTeX packages
https://bookdown.org/yihui/rmarkdown-cookbook/install-latex-pkgs.html
install.packages("tinytex")
tinytex::parse_install(text = "! LaTeX Error: File 'biblatex.sty' not found.")
tinytex::parse_install(text = "! LaTeX Error: File 'CJKutf8.sty' not found.")
tinytex::parse_install(text = "! LaTeX Error: File 'fancyhdr.sty' not found.")
tinytex::parse_install(text = "! LaTeX Error: File 'stmaryrd.sty' not found.")
tinytex::parse_install(text = "! LaTeX Error: File 'upgreek.sty' not found.")
tinytex::parse_install(text = "! LaTeX Error: File 'tikz.sty' not found.")
tinytex::parse_install(text = "! LaTeX Error: File 'tikz-cd.sty' not found.")
tinytex::parse_install(text = "! LaTeX Error: File 'adjustbox.sty' not found.")
tinytex::parse_install(text = "! LaTeX Error: File 'pgfplots.sty' not found.")
tinytex::parse_install(text = "! LaTeX Error: File 'units.sty' not found.")
tinytex::parse_install(text = "! LaTeX Error: File 'wasysym.sty' not found.")
tinytex::parse_install(text = "! LaTeX Error: File 'cancel.sty' not found.")
2 by 2
tinytex::parse_install(text = "! LaTeX Error: File 'biblatex.sty' not found. LaTeX Error: File 'CJKutf8.sty' not found.")
tinytex::parse_install(text = "! LaTeX Error: File 'fancyhdr.sty' not found. LaTeX Error: File 'stmaryrd.sty' not found.")
tinytex::parse_install(text = "! LaTeX Error: File 'upgreek.sty' not found. LaTeX Error: File 'tikz.sty' not found.")
tinytex::parse_install(text = "! LaTeX Error: File 'tikz-cd.sty' not found. LaTeX Error: File 'adjustbox.sty' not found.")
tinytex::parse_install(text = "! LaTeX Error: File 'pgfplots.sty' not found. LaTeX Error: File 'units.sty' not found.")
tinytex::parse_install(text = "! LaTeX Error: File 'wasysym.sty' not found. LaTeX Error: File 'cancel.sty' not found.")
https://github.com/muzimuzhi/thmtools GitHub: muzimuzhi / thmtools
'theorems-thmtools.module' moved to 'C:\Users\RW\AppData\Roaming\LyX2.4\layouts'
Dirac bra-ket notation = Dirac notation
https://tex.stackexchange.com/a/182985
My personal preference is to define keyboard shortcuts for the three items.
Go to Tools->Preferences->Editing->Shortcuts.

Function	Shortcut	Display
math-insert \scriptscriptstyle	Alt+Shift+S	$\scriptscriptstyle a$
math-insert \mathrm d	Alt+Shift+D	$\mathrm d$
math-insert \frac \mathrm d \mathrm d f	Alt+D D	$\frac{\mathrm d}{\mathrm d} f$
math-insert \frac \partial \partial f	Alt+D P	$\frac{\partial}{\partial} f$
math-insert \dfrac \partial \partial f	Alt+D O	$\dfrac{\partial}{\partial} f$
math-insert \int \mathrm d	Alt+I D	$\int \mathrm d$
math-insert \int \mathrm d \,	Alt+I F	$\int \mathrm d\,$
math-insert \left\langle | \right\rangle	Alt+Q Shift+|	$\left\langle \vert \right\rangle$
math-insert \left| \right\rangle	Alt+Q Shift+>	$\left\lvert \right\rangle$
math-insert \left\langle \right|	Alt+Q Shift+<	$\left\langle \right\rvert$
math-insert \left\langle \left| \right| \right\rangle	Alt+Q =	$\left\langle \left\lvert \right\rvert \right\rangle$
math-insert \right| \right\rangle \left\langle \left|	Alt+Q Backspace	$\rvert \rangle \langle \lvert$
math-insert \overset \shortrightarrow	Alt+Shift+>	$\overset{\shortrightarrow}{}$
math-insert \overset \shortleftarrow	Alt+Shift+<	$\overset{\shortleftarrow}{}$
math-insert \widetilde	Alt+Shift+~	$\widetilde{}$
math-insert \widehat	Alt+Shift+^	$\widehat{}$
math-insert \mathrm{i}	Alt+Shift+I	i
math-insert \mathrm{j}	Alt+Shift+J	j
math-insert \mathrm{k}	Alt+Shift+K	k
math-insert ^{\scriptscriptstyle \intercal}	Alt+Shift+T	$^{\scriptscriptstyle \intercal}$
math-insert ^{*}	Alt+Shift+*	*
math-insert ^{\scriptscriptstyle \dagger}	Alt+Shift+G	$^{\scriptscriptstyle \dagger}$
math-insert \cancel	Alt+Shift+C C	$\cancel{a}$
math-insert \cancelto	Alt+Shift+C T	$\cancelto{b}{a}$

https://tex.stackexchange.com/questions/659029/colour-packages-beyond-xcolor
base colors

4.1 Base colors (always available)

Figure 1: base colors

xcolor dvipsnames

This is how you get colored text:

\fcolorbox{blue}{yellow}{\textcolor{red}{Test}}: Test

\colorbox{yellow}{\textcolor{red}{Test}}: Test

\textcolor{red}{Test}: Test

These are the colors provided by the option dvipsnames/according to dvips' color.pro:

GreenYellow	Yellow	Goldenrod	Dandelion	Apricot
Peach	Melon	YellowOrange	Orange	BurntOrange
Bittersweet	RedOrange	Mahogany	Maroon	BrickRed
Red	OrangeRed	RubineRed	WildStrawberry	Salmon
CarnationPink	Magenta	VioletRed	Rhodamine	Mulberry
RedViolet	Fuchsia	Lavender	Thistle	Orchid
DarkOrchid	Purple	Plum	Violet	RoyalPurple
BlueViolet	Periwinkle	CadetBlue	CornflowerBlue	MidnightBlue
NavyBlue	RoyalBlue	Blue	Cerulean	Cyan
ProcessBlue	SkyBlue	Turquoise	TealBlue	Aquamarine
BlueGreen	Emerald	JungleGreen	SeaGreen	Green
ForestGreen	PineGreen	LimeGreen	YellowGreen	SpringGreen
OliveGreen	RawSienna	Sepia	Brown	Tan
Gray	Black	White		

Note that when using the package **color** instead of the package **xcolor**, besides the option **dvipsnames**, after **dvipsnames** you also need to specify the option **usenames**.

Note that document-classes like **beamer** and some packages internally load **xcolor** so that you need to specify options for **xcolor** already with the directives for loading these document-classes/packages.
Otherwise you might get errors about option-clashes and the like where with online-platforms like overleaf you need to view the so-called "raw log" for taking notice. The "raw-log" is the .log-file provided by the LaTeX-compiler itself while the LaTeX-compiler is running and what people asking for .log-file wish to see.

1

Figure 2: xcolor dvipsnames

https://en.wikibooks.org/wiki/LaTeX/Colors

演算法 0.1 recolor = RegEx coloring via expl3 for LaTeX

```
% New
\ExplSyntaxOn
\NewDocumentCommand{\recolor}{m}
{
\tl_set:Nn \l_tmpa_tl { #1 }
% \widetilde{v}, v
\regex_replace_all:nnN
{ \c{widetilde}{v} }
{ 0001 }
\l_tmpa_tl
% add \b before & after single v or else easily replaced to cause errors
\regex_replace_all:nnN
{ \b v \b }
{ {\c{color}{orange}{\0}} }
\l_tmpa_tl
\regex_replace_all:nnN
{ 0001 }
{ {\c{color}{red!50}{\c{widetilde}{v}}} }
\l_tmpa_tl
\tl_use:N \l_tmpa_tl
}
\ExplSyntaxOff
```

tensor

Joey Yu Hsu, M.D.
at NTUH, since 2024/12

January 19, 2025

Chapter 1

tensor algebra

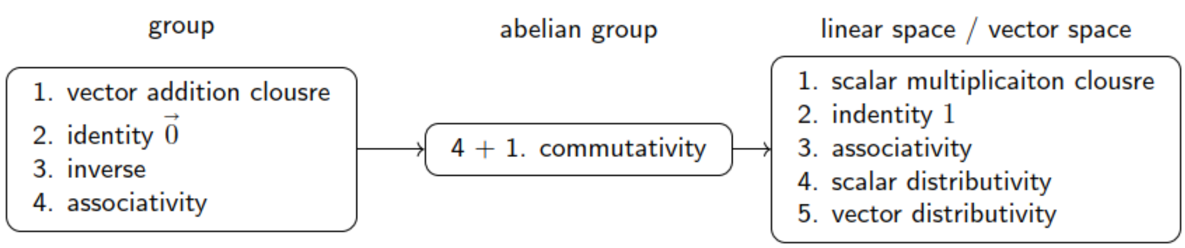

Figure 1.1: vector space axiom construction

vector space = linear space

$\mathcal{V}$ is a vector space

$$\Leftrightarrow \mathcal{V} = \mathcal{V}(\mathbb{F},+,\cdot) = (\mathcal{V},\mathbb{F},+,\cdot) \text{ is a vector space over the field } \mathbb{F} : \begin{cases} + & \text{additivity} \\ \cdot & \text{homogeneity} \end{cases}$$

$$\Leftrightarrow \begin{cases} \mathbb{F} \text{ is a field} & (f)\text{ field} \\ \mathcal{V} \neq \emptyset & (ne)\text{ nonempty set} \\ \mathcal{V} = (\mathcal{V},+) \text{ is a commutative group} \Leftrightarrow \mathcal{V} = (\mathcal{V},+) \text{ is an abelian group} & (va)\text{ vector addition} \\ \cdot : \mathbb{F}\times\mathcal{V} \dot{\to} \mathcal{V} \Leftrightarrow \forall s\in\mathbb{F}, \forall\vec{v}\in\mathcal{V}, \exists\vec{u}\in\mathcal{V}\left[\vec{u} = s\vec{v} = s\cdot\vec{v}\right] & (sm)\text{ scalar multiplication} \\ \begin{cases} \exists 1\in\mathbb{F}, \forall\vec{v}\in\mathcal{V}\left[1\vec{v} = \vec{v}\right] & (e)\text{ identity} \\ \forall s,t\in\mathbb{F}, \vec{v}\in\mathcal{V}\left[s\left(t\vec{v}\right) = (st)\,\vec{v}\right] & (a)\text{ associativity} \\ \forall s,t\in\mathbb{F}, \vec{v}\in\mathcal{V}\left[(s+t)\,\vec{v} = s\vec{v}+t\vec{v}\right] & (ds)\text{ scalar distributivity} \\ \forall s\in\mathbb{F}, \vec{u},\vec{v}\in\mathcal{V}\left[s\left(\vec{u}+\vec{v}\right) = s\vec{u}+s\vec{v}\right] & (dv)\text{ vector distributivity} \end{cases} & (sm)\text{ axioms} \end{cases}$$

$$\Leftrightarrow \begin{cases} \mathbb{F} \text{ is a field} & (f)\text{ field} \\ \mathcal{V} \neq \emptyset & (ne)\text{ nonempty set} \\ \begin{cases} \mathcal{V} = (\mathcal{V},+) = (\mathcal{V},+_{\mathcal{V}}) \text{ is a group} & (g)\text{ group} \\ \forall\vec{u},\vec{v}\in\mathcal{V}\left[\vec{u}+\vec{v} = \vec{v}+\vec{u}\right] & (c)\text{ commutativity} \end{cases} & (va)\text{ vector addition} \\ \cdot = \cdot_{\mathbb{F}\times\mathcal{V}} : \mathbb{F}\times\mathcal{V} \dot{\to} \mathcal{V} \Leftrightarrow \forall s\in\mathbb{F}, \forall\vec{v}\in\mathcal{V}, \exists\vec{u}\in\mathcal{V}\left[\vec{u} = s\vec{v} = s\cdot\vec{v}\right] & (sm)\text{ scalar multiplication} \\ \begin{cases} \exists 1\in\mathbb{F}, \forall\vec{v}\in\mathcal{V}\left[1\vec{v} = \vec{v}\right] & (e)\text{ identity} \\ \forall s,t\in\mathbb{F}, \vec{v}\in\mathcal{V}\left[s\left(t\vec{v}\right) = (st)\,\vec{v}\right] & (a)\text{ associativity} \\ \forall s,t\in\mathbb{F}, \vec{v}\in\mathcal{V}\left[(s+t)\,\vec{v} = s\vec{v}+t\vec{v}\right] & (ds)\text{ scalar distributivity} \\ \forall s\in\mathbb{F}, \vec{u},\vec{v}\in\mathcal{V}\left[s\left(\vec{u}+\vec{v}\right) = s\vec{u}+s\vec{v}\right] & (dv)\text{ vector distributivity} \end{cases} & (sm)\text{ axioms} \end{cases}$$

$$\Leftrightarrow \begin{cases} \mathbb{F} = \mathbb{F}(+_{\mathbb{F}},\cdot_{\mathbb{F}}) = (\mathbb{F},+_{\mathbb{F}},\cdot_{\mathbb{F}}) = (\mathbb{F},+,\cdot) \text{ is a field} & (f) \\ \mathcal{V}\neq\emptyset & (ne) \\ \begin{cases} \begin{cases} + : \mathcal{V}\times\mathcal{V} = \mathcal{V}^2 \overset{+}{\to} \mathcal{V} \Leftrightarrow \forall\vec{u},\vec{v}\in\mathcal{V}, \exists!\vec{w}\in\mathcal{V}\left[\vec{w} = \vec{u}+\vec{v}\right] & (cl)\text{ closure} \\ \exists!\vec{0}\in\mathcal{V}, \forall\vec{v}\in\mathcal{V}\left[\vec{0}+\vec{v} = \vec{v}\right] & (e)\text{ identity} \\ \forall\vec{v}\in\mathcal{V}, \exists! -\vec{v}\in\mathcal{V}\left[\left(-\vec{v}\right)+\vec{v} = \vec{0}\right] & (i)\text{ inverse} \\ \forall\vec{u},\vec{v},\vec{w}\in\mathcal{V}\left[\vec{u}+\left(\vec{v}+\vec{w}\right) = \left(\vec{u}+\vec{v}\right)+\vec{w}\right] & (a)\text{ associativity} \end{cases} & (g) \\ \forall\vec{u},\vec{v}\in\mathcal{V}\left[\vec{u}+\vec{v} = \vec{v}+\vec{u}\right] & (c) \end{cases} & (va) \\ \begin{cases} \cdot : \mathbb{F}\times\mathcal{V} \dot{\to} \mathcal{V} \Leftrightarrow \forall s\in\mathbb{F}, \forall\vec{v}\in\mathcal{V}, \exists!\vec{u}\in\mathcal{V}\left[\vec{u} = s\vec{v} = s\cdot\vec{v}\right] & (cl)\text{ closure} \\ \exists!1\in\mathbb{F}, \forall\vec{v}\in\mathcal{V}\left[1\vec{v} = \vec{v}\right] & (e)\text{ identity} \\ \forall s,t\in\mathbb{F}, \vec{v}\in\mathcal{V}\left[s\left(t\vec{v}\right) = s\cdot_{\mathbb{F}\times\mathcal{V}}\left(t\cdot_{\mathbb{F}\times\mathcal{V}}\vec{v}\right) = (s\cdot_{\mathbb{F}}t)\cdot_{\mathbb{F}\times\mathcal{V}}\vec{v} = (st)\,\vec{v}\right] & (a)\text{ associativity} \\ \forall s,t\in\mathbb{F}, \vec{v}\in\mathcal{V}\left[(s+t)\,\vec{v} = (s+_{\mathbb{F}}t)\,\vec{v} = s\vec{v}+_{\mathcal{V}}t\vec{v} = s\vec{v}+t\vec{v}\right] & (ds)\text{ scalar distributivity} \\ \forall s\in\mathbb{F}, \vec{u},\vec{v}\in\mathcal{V}\left[s\left(\vec{u}+\vec{v}\right) = s\vec{u}+s\vec{v}\right] & (dv)\text{ vector distributivity} \end{cases} & (sm) \end{cases}$$

linear dependency

$$\left\{\vec{v}_i\right\}_i \text{ is linearly dependent}$$

$$\Leftrightarrow \exists c^i \in \mathbb{F}\left[\left(\cdots, c^i, \cdots\right) \neq (\cdots, 0, \cdots) \wedge \vec{v}_i c^i = \vec{0}\right] \qquad \vec{v}_i c^i = \sum_i \vec{v}_i c^i$$

$$\Leftrightarrow \exists c^i \in \mathbb{F}\left[\neg\left[\left(\cdots, c^i, \cdots\right) = (\cdots, 0, \cdots)\right] \wedge \left[\vec{v}_i c^i = \vec{0}\right]\right] \qquad \text{E.s.c.} = \text{Einstein summation convention}$$

$$\Leftrightarrow \exists c^i \in \mathbb{F}\left[\neg\left[\left(\cdots, c^i, \cdots\right) = (\cdots, 0, \cdots)\right] \wedge \neg\left[\vec{v}_i c^i \neq \vec{0}\right]\right]$$

$$\Leftrightarrow \exists c^i \in \mathbb{F}\left[\neg\left[\left(\cdots, c^i, \cdots\right) = (\cdots, 0, \cdots) \vee \vec{v}_i c^i \neq \vec{0}\right]\right]$$

$$\Leftrightarrow \exists c^i \in \mathbb{F}\left[\neg\left[\left(\cdots, c^i, \cdots\right) = (\cdots, 0, \cdots) \vee \neg\left[\vec{v}_i c^i = \vec{0}\right]\right]\right]$$

$$\Leftrightarrow \exists c^i \in \mathbb{F}\left[\neg\left[\neg\left[\vec{v}_i c^i = \vec{0}\right] \vee \left(\cdots, c^i, \cdots\right) = (\cdots, 0, \cdots)\right]\right]$$

$$\Leftrightarrow \exists c^i \in \mathbb{F} \neg\left[\vec{v}_i c^i = \vec{0} \Rightarrow \left(\cdots, c^i, \cdots\right) = (\cdots, 0, \cdots)\right]$$

$$\Leftrightarrow \neg\forall c^i \in \mathbb{F}\left[\vec{v}_i c^i = \vec{0} \Rightarrow \left(\cdots, c^i, \cdots\right) = (\cdots, 0, \cdots)\right]$$

$$\left\{\vec{e}_i\right\}_i \text{ is linearly independent}$$

$$\Leftrightarrow \neg\exists v^i \in \mathbb{F}\left[\left(\cdots, v^i, \cdots\right) \neq (\cdots, 0, \cdots) \wedge \vec{e}_i v^i = \vec{0}\right] \qquad \vec{e}_i v^i = \sum_i \vec{e}_i v^i$$

$$\Leftrightarrow \neg\neg\forall v^i \in \mathbb{F}\left[\vec{e}_i v^i = \vec{0} \Rightarrow \left(\cdots, v^i, \cdots\right) = (\cdots, 0, \cdots)\right] \qquad \text{E.s.c.} = \text{Einstein summation convention}$$

$$\Leftrightarrow \forall v^i \in \mathbb{F}\left[\vec{e}_i v^i = \vec{0} \Rightarrow \left(\cdots, v^i, \cdots\right) = (\cdots, 0, \cdots)\right]$$

dimension

$$\dim \mathcal{V} = \underset{n}{\arg\max}\left|\left\{\vec{e}_i \middle| \vec{e}_i v^i = \vec{0} \Rightarrow \left(\cdots, v^i, \cdots\right) = (\cdots, 0, \cdots)\right\}_{i=1}^{n}\right|$$

linearity

$$\phi \text{ is linear}$$

$$\Leftrightarrow \begin{cases} \phi : D \to C & \Leftrightarrow \phi \in C^D \\ \phi\left(s_x x + s_y y\right) = s_x \phi(x) + s_y \phi(y) & \forall s_x x, s_y y \in D, \exists s_x \phi(x), s_y \phi(y) \in C \end{cases}$$

antilinearity

$$\psi \text{ is antilinear}$$

$$\Leftrightarrow \begin{cases} \psi : D \to C & \Leftrightarrow \phi \in C^D \\ s_x^* \text{ is a conjugate of } s_x & \forall s_x \\ \psi\left(s_x x + s_y y\right) = s_x^* \psi(x) + s_y^* \psi(y) & \forall s_x x, s_y y \in D, \exists s_x^* \psi(x), s_y^* \psi(y) \in C \end{cases}$$

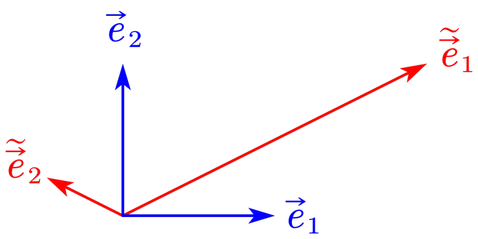

Figure 1.2: old basis, new basis

vector basis = vector space old basis = vector space $\mathcal{V}$ old basis $\mathfrak{E}$

$$\mathfrak{E} = \left\{\vec{e}_i\right\}_{i=1}^{2} = \left\{\vec{e}_1, \vec{e}_2\right\} \subset \mathcal{V} = \operatorname{span}\left\{\vec{e}_1, \vec{e}_2\right\} = \operatorname{span}\left\{\vec{e}_i\right\}_{i=1}^{2} = \operatorname{span}\mathfrak{E}$$

$$\dim \mathcal{V} = |\mathfrak{E}| = \left|\left\{\vec{e}_i\right\}_{i=1}^{2}\right| = \left|\left\{\vec{e}_1, \vec{e}_2\right\}\right| = 2 = \left|\left\{\vec{e}_i\right\}_{i=1}^{\dim \mathcal{V}}\right|$$

vector space new basis = vector space $\mathcal{V}$ new basis $\widetilde{\mathfrak{E}}$

$$\widetilde{\mathfrak{E}} = \left\{\tilde{\vec{e}}_i\right\}_{i=1}^{2} = \left\{\tilde{\vec{e}}_1, \tilde{\vec{e}}_2\right\} \subset \mathcal{V} = \operatorname{span}\left\{\tilde{\vec{e}}_1, \tilde{\vec{e}}_2\right\} = \operatorname{span}\left\{\tilde{\vec{e}}_i\right\}_{i=1}^{2} = \operatorname{span}\widetilde{\mathfrak{E}}$$

$$\dim\mathcal{V} = \left|\widetilde{\mathfrak{E}}\right| = \left|\left\{\tilde{\vec{e}}_i\right\}_{i=1}^{2}\right| = \left|\left\{\tilde{\vec{e}}_1, \tilde{\vec{e}}_2\right\}\right| = 2 = \left|\left\{\tilde{\vec{e}}_i\right\}_{i=1}^{\dim\mathcal{V}}\right|$$

vector basis = vector space old basis = vector space $\mathcal{V}$ old basis $\mathfrak{E}$

$$\mathfrak{E} = \vec{e} = \vec{e}_i = \left\{\vec{e}_i\right\} = \left\{\vec{e}_i\right\}_{i=1}^{\dim\mathcal{V}} = \left\{\vec{e}_i\right\}_{i=1}^{2} = \left\{\vec{e}_1, \vec{e}_2\right\}$$

vector basis row vector = vector space old basis row vector = vector space $\mathcal{V}$ old basis row vector E

$$E = \left[\vec{e}\right] = \left[\vec{e}_i\right] = \left[\vec{e}_i\right]_{i=1}^{\dim\mathcal{V}} = \left[\vec{e}_i\right]_{i=1}^{2} = \left[\vec{e}_i\right]_{1\times 2} = \begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}$$

vector basis = vector space new basis = vector space $\mathcal{V}$ new basis $\widetilde{\mathfrak{E}}$

$$\widetilde{\mathfrak{E}} = \tilde{\vec{e}} = \tilde{\vec{e}}_i = \left\{\tilde{\vec{e}}_i\right\} = \left\{\tilde{\vec{e}}_i\right\}_{i=1}^{\dim\mathcal{V}} = \left\{\tilde{\vec{e}}_i\right\}_{i=1}^{2} = \left\{\tilde{\vec{e}}_1, \tilde{\vec{e}}_2\right\}$$

vector basis row vector = vector space new basis row vector = vector space $\mathcal{V}$ new basis row vector $\widetilde{E}$

$$\widetilde{E} = \left[\tilde{\vec{e}}\right] = \left[\tilde{\vec{e}}_i\right] = \left[\tilde{\vec{e}}_i\right]_{i=1}^{\dim\mathcal{V}} = \left[\tilde{\vec{e}}_i\right]_{i=1}^{2} = \left[\tilde{\vec{e}}_i\right]_{1\times 2} = \begin{bmatrix}\tilde{\vec{e}}_1 & \tilde{\vec{e}}_2\end{bmatrix}$$

forward transformation F = vector space $\mathcal{V}$ new basis row vector $\widetilde{E}$ represented by old basis row vector E

$$\tilde{\vec{e}}_1 = \vec{e}_1\tilde{e}_1^1 + \vec{e}_2\tilde{e}_1^2 = \begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\begin{bmatrix}\tilde{e}_1^1 \\ \tilde{e}_1^2\end{bmatrix} = \left[\vec{e}\right]\begin{bmatrix}\tilde{e}_1^1 \\ \tilde{e}_1^2\end{bmatrix}_{\vec{e}} = \left[\vec{e}\right]\left[\tilde{e}_1\right]_{\vec{e}} = E\left[\tilde{e}_1\right]_E$$

$\left[\tilde{e}_1\right]_E = \left[\tilde{e}_2\right]_{\vec{e}}$: coordinate vector of vector $\tilde{\vec{e}}_1$ relative to basis $E = \left[\vec{e}\right]$

$$\tilde{\vec{e}}_2 = \vec{e}_1\tilde{e}_2^1 + \vec{e}_2\tilde{e}_2^2 = \begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\begin{bmatrix}\tilde{e}_2^1 \\ \tilde{e}_2^2\end{bmatrix} = \left[\vec{e}\right]\begin{bmatrix}\tilde{e}_2^1 \\ \tilde{e}_2^2\end{bmatrix}_{\vec{e}} = \left[\vec{e}\right]\left[\tilde{e}_2\right]_{\vec{e}} = E\left[\tilde{e}_2\right]_E$$

$$\widetilde{E} = \left[\tilde{\vec{e}}\right] = \begin{bmatrix}\tilde{\vec{e}}_1 & \tilde{\vec{e}}_2\end{bmatrix} = \begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\begin{bmatrix}\tilde{e}_1^1 & \tilde{e}_2^1 \\ \tilde{e}_1^2 & \tilde{e}_2^2\end{bmatrix} = \left[\vec{e}\right]\left[\tilde{e}\right]_{\vec{e}} = E\left[\widetilde{E}\right]_E = EF$$

$$\widetilde{E} = \left[\tilde{\vec{e}}\right] = \left[\vec{e}\right]\left[\tilde{e}\right]_{\vec{e}} = E\left[\widetilde{E}\right]_E = EF$$

$$\widetilde{E} = \left[\tilde{\vec{e}}\right] = \left[\tilde{\vec{e}}_j\right]_{1\times n} = \left[\vec{e}_i\right]_{1\times n}\left[\tilde{e}_j^i\right]_n^{n\times} = \left[\vec{e}\right]\left[\tilde{e}\right]_{\vec{e}} = E\left[\widetilde{E}\right]_E = EF \qquad n = \dim\mathcal{V} \Rightarrow \left[\widetilde{E}\right]_E = \left[\tilde{e}_j^i\right]$$

$$\left[\tilde{\vec{e}}_j\right]_{1\times n} = \left[\vec{e}_i\right]_{1\times n}\left[\tilde{e}_j^i\right]_n^{n\times} = \left[\vec{e}_i\tilde{e}_j^i\right]_{1\times n} = \left[\vec{e}_iF_j^i\right]_{1\times n} = \left[\vec{e}_i\right]_{1\times n}\left[F_j^i\right]_n^{n\times}$$

$$\left[\tilde{\vec{e}}_j\right] = \left[\vec{e}_i\right]\left[\tilde{e}_j^i\right] = \left[\vec{e}_i\tilde{e}_j^i\right] = \left[\vec{e}_iF_j^i\right] = \left[\vec{e}_i\right]\left[F_j^i\right]$$

$$\tilde{\vec{e}}_j = \vec{e}_i\tilde{e}_j^i = \vec{e}_iF_j^i$$

$$\tilde{\vec{e}}_j = \vec{e}_i\tilde{e}_j^i = \sum_i \vec{e}_i\tilde{e}_j^i = \sum_{i=1}^{\dim\mathcal{V}} \vec{e}_i\tilde{e}_j^i = \sum_{i=1}^{n} \vec{e}_i\tilde{e}_j^i \qquad \text{E.s.c. = Einstein summation convention = Einstein notation}$$

$$F = \left[\widetilde{E}\right]_E = \left[\tilde{e}_j^i\right] = \left[\tilde{e}\right]_{\vec{e}} \overset{\text{e.g.}}{=} \begin{bmatrix}\tilde{e}_1^1 & \tilde{e}_2^1 \\ \tilde{e}_1^2 & \tilde{e}_2^2\end{bmatrix}$$

$\left[\tilde{e}\right]_{\vec{e}} = \left[\widetilde{E}\right]_E$: coordinate vector matrix of row vector $\widetilde{E} = \left[\tilde{\vec{e}}_j\right]_{1\times n}$ relative to basis $E = \left[\vec{e}\right]$

$$\widetilde{E} = \left[\tilde{\vec{e}}\right] = \left[\vec{e}\right]\left[\tilde{e}\right]_{\vec{e}} = E\left[\widetilde{E}\right]_E = EF$$

backward transformation B = vector space $\mathcal{V}$ new basis row vector E represented by old basis row vector $\widetilde{E}$

$$\vec{e}_1 = \tilde{\vec{e}}_1 e_1^1 + \tilde{\vec{e}}_2 e_1^2 = \begin{bmatrix} \tilde{\vec{e}}_1 & \tilde{\vec{e}}_2 \end{bmatrix} \begin{bmatrix} e_1^1 \\ e_1^2 \end{bmatrix} = \begin{bmatrix} \tilde{\vec{e}} \end{bmatrix} \begin{bmatrix} e_1^1 \\ e_1^2 \end{bmatrix}_{\tilde{\vec{e}}} = \begin{bmatrix} \tilde{\vec{e}} \end{bmatrix} [e_1]_{\tilde{\vec{e}}} = \widetilde{E} [e_1]_{\widetilde{E}}$$

$[e_1]_{\widetilde{E}} = [e_1]_{\tilde{\vec{e}}}$: coordinate vector of vector $\vec{e}_1$ relative to basis $\widetilde{E} = \begin{bmatrix} \tilde{\vec{e}} \end{bmatrix}$

$$\vec{e}_2 = \tilde{\vec{e}}_1 e_2^1 + \tilde{\vec{e}}_2 e_2^2 = \begin{bmatrix} \tilde{\vec{e}}_1 & \tilde{\vec{e}}_2 \end{bmatrix} \begin{bmatrix} e_2^1 \\ e_2^2 \end{bmatrix} = \begin{bmatrix} \tilde{\vec{e}} \end{bmatrix} \begin{bmatrix} e_2^1 \\ e_2^2 \end{bmatrix}_{\tilde{\vec{e}}} = \begin{bmatrix} \tilde{\vec{e}} \end{bmatrix} [e_2]_{\tilde{\vec{e}}} = \widetilde{E} [e_2]_{\widetilde{E}}$$

$$E = \begin{bmatrix} \vec{e} \end{bmatrix} = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix} = \begin{bmatrix} \tilde{\vec{e}}_1 & \tilde{\vec{e}}_2 \end{bmatrix} \begin{bmatrix} e_1^1 & e_2^1 \\ e_1^2 & e_2^2 \end{bmatrix} = \begin{bmatrix} \tilde{\vec{e}} \end{bmatrix} [e]_{\tilde{\vec{e}}} = \widetilde{E} [E]_{\widetilde{E}} = \widetilde{E} B$$

$$E = \begin{bmatrix} \vec{e} \end{bmatrix} = \begin{bmatrix} \tilde{\vec{e}} \end{bmatrix} [e]_{\tilde{\vec{e}}} = \widetilde{E} [E]_{\widetilde{E}} = \widetilde{E} B$$

$$E = \begin{bmatrix} \vec{e} \end{bmatrix} = \begin{bmatrix} \vec{e}_j \end{bmatrix}_{1\times n} = \begin{bmatrix} \tilde{\vec{e}}_i \end{bmatrix}_{1\times n} \left[e_j^i\right]_n^{n\times} = \begin{bmatrix} \tilde{\vec{e}} \end{bmatrix} [e]_{\tilde{\vec{e}}} = \widetilde{E} [E]_{\widetilde{E}} = \widetilde{E} B \qquad n = \dim \mathcal{V} \Rightarrow [E]_{\widetilde{E}} = \left[e_j^i\right]$$

$$\begin{bmatrix} \vec{e}_j \end{bmatrix}_{1\times n} = \begin{bmatrix} \tilde{\vec{e}}_i \end{bmatrix}_{1\times n} \left[e_j^i\right]_n^{n\times} = \begin{bmatrix} \tilde{\vec{e}}_i e_j^i \end{bmatrix}_{1\times n} = \begin{bmatrix} \tilde{\vec{e}}_i B_j^i \end{bmatrix}_{1\times n} = \begin{bmatrix} \tilde{\vec{e}}_i \end{bmatrix}_{1\times n} \left[B_j^i\right]_n^{n\times}$$

$$\begin{bmatrix} \vec{e}_j \end{bmatrix} = \begin{bmatrix} \tilde{\vec{e}}_i \end{bmatrix} \left[e_j^i\right] = \begin{bmatrix} \tilde{\vec{e}}_i e_j^i \end{bmatrix} = \begin{bmatrix} \tilde{\vec{e}}_i B_j^i \end{bmatrix} = \begin{bmatrix} \tilde{\vec{e}}_i \end{bmatrix} \left[B_j^i\right]$$

$$\vec{e}_j = \tilde{\vec{e}}_i e_j^i = \tilde{\vec{e}}_i B_j^i$$

$$\vec{e}_j = \tilde{\vec{e}}_i e_j^i = \sum_i \tilde{\vec{e}}_i e_j^i = \sum_{i=1}^{\dim \mathcal{V}} \tilde{\vec{e}}_i e_j^i = \sum_{i=1}^{n} \tilde{\vec{e}}_i e_j^i \qquad \text{E.s.c. = Einstein summation convention = Einstein notation}$$

$$B = [E]_{\widetilde{E}} = \left[e_j^i\right] = [e]_{\tilde{\vec{e}}} \overset{\text{e.g.}}{=} \begin{bmatrix} e_1^1 & e_2^1 \\ e_1^2 & e_2^2 \end{bmatrix}$$

$[e]_{\tilde{\vec{e}}} = [E]_{\widetilde{E}}$: coordinate vector matrix of row vector $E = \begin{bmatrix} \vec{e}_j \end{bmatrix}_{1\times n}$ relative to basis $\widetilde{E} = \begin{bmatrix} \tilde{\vec{e}} \end{bmatrix}$

$$E = \begin{bmatrix} \vec{e} \end{bmatrix} = \begin{bmatrix} \tilde{\vec{e}} \end{bmatrix} [e]_{\tilde{\vec{e}}} = \widetilde{E} [E]_{\widetilde{E}} = \widetilde{E} B$$

forward transformation F and backward transformation B are inverse matrix(ces) of each other

$$\widetilde{E} = \begin{bmatrix} \tilde{\vec{e}} \end{bmatrix} = \begin{bmatrix} \vec{e} \end{bmatrix} [\tilde{e}]_{\vec{e}} = E \begin{bmatrix} \widetilde{E} \end{bmatrix}_E = EF$$

$$E = \begin{bmatrix} \vec{e} \end{bmatrix} = \begin{bmatrix} \tilde{\vec{e}} \end{bmatrix} [e]_{\tilde{\vec{e}}} = \widetilde{E} [E]_{\widetilde{E}} = \widetilde{E} B$$

$$\begin{aligned} \widetilde{E} = \begin{bmatrix} \tilde{\vec{e}} \end{bmatrix} &= \begin{bmatrix} \vec{e} \end{bmatrix} [\tilde{e}]_{\vec{e}} = E \begin{bmatrix} \widetilde{E} \end{bmatrix}_E = EF \\ &= \begin{bmatrix} \tilde{\vec{e}} \end{bmatrix} [e]_{\tilde{\vec{e}}} [\tilde{e}]_{\vec{e}} = \widetilde{E} [E]_{\widetilde{E}} \begin{bmatrix} \widetilde{E} \end{bmatrix}_E = \widetilde{E} BF \qquad E = \begin{bmatrix} \vec{e} \end{bmatrix} = \begin{bmatrix} \tilde{\vec{e}} \end{bmatrix} [e]_{\tilde{\vec{e}}} = \widetilde{E} [E]_{\widetilde{E}} = \widetilde{E} B \\ \cancel{\widetilde{E}} = \cancel{\begin{bmatrix} \tilde{\vec{e}} \end{bmatrix}} &= \cancel{\begin{bmatrix} \tilde{\vec{e}} \end{bmatrix}} [e]_{\tilde{\vec{e}}} [\tilde{e}]_{\vec{e}} = \cancel{\widetilde{E}} [E]_{\widetilde{E}} \begin{bmatrix} \widetilde{E} \end{bmatrix}_E = \cancel{\widetilde{E}} BF \\ 1 &= [e]_{\tilde{\vec{e}}} [\tilde{e}]_{\vec{e}} = [E]_{\widetilde{E}} \begin{bmatrix} \widetilde{E} \end{bmatrix}_E = BF \end{aligned}$$

$$\begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} = \delta_j^i = \left[\delta_j^i\right] = \left[e_k^i \tilde{e}_j^k\right] = [e]_{\tilde{\vec{e}}} [\tilde{e}]_{\vec{e}} = \left[e_k^i\right] \left[\tilde{e}_j^k\right] = [E]_{\widetilde{E}} \begin{bmatrix} \widetilde{E} \end{bmatrix}_E = BF \qquad \begin{aligned} \begin{bmatrix} \widetilde{E} \end{bmatrix}_E &= \left[\tilde{e}_j^i\right] = \left[\tilde{e}_k^i\right] \\ [E]_{\widetilde{E}} &= \left[e_j^i\right] = \left[e_j^k\right] \end{aligned}$$

$$E = \left[\vec{e}\right] = \left[\vec{\tilde{e}}\right] \left[e\right]_{\vec{\tilde{e}}} = \tilde{E} \left[E\right]_{\tilde{E}} = \tilde{E} B$$

$$= \left[\vec{e}\right] \left[\tilde{e}\right]_{\vec{e}} \left[e\right]_{\vec{\tilde{e}}} = E \left[\tilde{E}\right]_{E} \left[E\right]_{\tilde{E}} = EFB \qquad \tilde{E} = \left[\vec{\tilde{e}}\right] = \left[\vec{e}\right] \left[\tilde{e}\right]_{\vec{e}} = E \left[\tilde{E}\right]_{E} = EF$$

$$\cancel{E} = \left[\cancel{\vec{e}}\right] = \left[\cancel{\vec{e}}\right] \left[\tilde{e}\right]_{\vec{e}} \left[e\right]_{\vec{\tilde{e}}} = \cancel{E} \left[\tilde{E}\right]_{E} \left[E\right]_{\tilde{E}} = \cancel{E} FB$$

$$1 = \left[\tilde{e}\right]_{\vec{e}} \left[e\right]_{\vec{\tilde{e}}} = \left[\tilde{E}\right]_{E} \left[E\right]_{\tilde{E}} = FB$$

$$\begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} = \delta^i_j = \left[\delta^i_j\right] = \left[\tilde{e}^i_k e^k_j\right] = \left[\tilde{e}\right]_{\vec{e}} \left[e\right]_{\vec{\tilde{e}}} = \left[\tilde{e}^i_k\right] \left[e^k_j\right] = \left[\tilde{E}\right]_{E} \left[E\right]_{\tilde{E}} = FB \qquad \begin{matrix} \left[\tilde{E}\right]_{E} = \left[\tilde{e}^i_j\right] = \left[\tilde{e}^k_j\right] \\ \left[E\right]_{\tilde{E}} = \left[e^i_j\right] = \left[e^i_k\right] \end{matrix}$$

invariant vector object $\vec{v}$ but variant vector component scalar

$$\boldsymbol{v} = \vec{v} \in \mathcal{V} = \operatorname{span}\left\{\vec{e}_1, \vec{e}_2\right\} = \operatorname{span}\left\{\vec{e}_i\right\}_{i=1}^{\dim \mathcal{V}=2} = \operatorname{span}\mathfrak{E} = \operatorname{span}\left\{\vec{\tilde{e}}_1, \vec{\tilde{e}}_2\right\} = \operatorname{span}\left\{\vec{\tilde{e}}_i\right\}_{i=1}^{\dim \mathcal{V}=2} = \operatorname{span}\tilde{\mathfrak{E}}$$

vector decomposition over vector space basis

$$\vec{v} = \vec{e}_1 v^1 + \vec{e}_2 v^2 = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \left[\vec{e}\right] \begin{bmatrix} v^1 \\ v^2 \end{bmatrix}_{\vec{e}} = \left[\vec{e}\right] \left[v\right]_{\vec{e}} = E \left[v\right]_{E}$$

$$\vec{v} = \vec{\tilde{e}}_1 \tilde{v}^1 + \vec{\tilde{e}}_2 \tilde{v}^2 = \begin{bmatrix} \vec{\tilde{e}}_1 & \vec{\tilde{e}}_2 \end{bmatrix} \begin{bmatrix} \tilde{v}^1 \\ \tilde{v}^2 \end{bmatrix} = \left[\vec{\tilde{e}}\right] \begin{bmatrix} \tilde{v}^1 \\ \tilde{v}^2 \end{bmatrix}_{\vec{\tilde{e}}} = \left[\vec{\tilde{e}}\right] \left[v\right]_{\vec{\tilde{e}}} = \tilde{E} \left[v\right]_{\tilde{E}}$$

$$\vec{v} = \left[\vec{e}\right] \left[v\right]_{\vec{e}} = E \left[v\right]_{E}$$

$$= \left[\vec{\tilde{e}}\right] \left[e\right]_{\vec{\tilde{e}}} \left[v\right]_{\vec{e}} = \tilde{E} \left[E\right]_{\tilde{E}} \left[v\right]_{E} = \tilde{E} B \left[v\right]_{E} \qquad E = \left[\vec{e}\right] = \left[\vec{\tilde{e}}\right] \left[e\right]_{\vec{\tilde{e}}} = \tilde{E} \left[E\right]_{\tilde{E}} = \tilde{E} B$$

$$= \vec{v} = \left[\vec{\tilde{e}}\right] \left[v\right]_{\vec{\tilde{e}}} = \tilde{E} \left[v\right]_{\tilde{E}}$$

$$= \left[\vec{e}\right] \left[\tilde{e}\right]_{\vec{e}} \left[v\right]_{\vec{\tilde{e}}} = E \left[\tilde{E}\right]_{E} \left[v\right]_{\tilde{E}} = EF \left[v\right]_{\tilde{E}} \qquad \tilde{E} = \left[\vec{\tilde{e}}\right] = \left[\vec{e}\right] \left[\tilde{e}\right]_{\vec{e}} = E \left[\tilde{E}\right]_{E} = EF$$

$$E \left[v\right]_{E} = \vec{v} = EF \left[v\right]_{\tilde{E}} \qquad \Rightarrow E \left[v\right]_{E} = EF \left[v\right]_{\tilde{E}}$$

$$\tilde{E} \left[v\right]_{\tilde{E}} = \vec{v} = \tilde{E} B \left[v\right]_{E} \qquad \Rightarrow \tilde{E} \left[v\right]_{\tilde{E}} = \tilde{E} B \left[v\right]_{E}$$

$$\cancel{E} \left[v\right]_{E} = \cancel{E} F \left[v\right]_{\tilde{E}} \qquad \Rightarrow \left[v\right]_{E} = F \left[v\right]_{\tilde{E}}$$

$$\cancel{\tilde{E}} \left[v\right]_{\tilde{E}} = \cancel{\tilde{E}} B \left[v\right]_{E} \qquad \Rightarrow \left[v\right]_{\tilde{E}} = B \left[v\right]_{E}$$

$$\begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \overset{\text{e.g.}}{=} \left[v\right]_{E} = F \left[v\right]_{\tilde{E}} = \left[\tilde{E}\right]_{E} \left[v\right]_{\tilde{E}} = \left[\tilde{e}\right]_{\vec{e}} \left[v\right]_{\vec{\tilde{e}}} \overset{\text{e.g.}}{=} \begin{bmatrix} \tilde{e}^1_1 & \tilde{e}^1_2 \\ \tilde{e}^2_1 & \tilde{e}^2_2 \end{bmatrix} \begin{bmatrix} \tilde{v}^1 \\ \tilde{v}^2 \end{bmatrix} \qquad F = \left[\tilde{E}\right]_{E} = \left[\tilde{e}\right]_{\vec{e}} \overset{\text{e.g.}}{=} \begin{bmatrix} \tilde{e}^1_1 & \tilde{e}^1_2 \\ \tilde{e}^2_1 & \tilde{e}^2_2 \end{bmatrix}$$

$$\begin{bmatrix} \tilde{v}^1 \\ \tilde{v}^2 \end{bmatrix} \overset{\text{e.g.}}{=} \left[v\right]_{\tilde{E}} = B \left[v\right]_{E} = \left[E\right]_{\tilde{E}} \left[v\right]_{E} = \left[e\right]_{\vec{\tilde{e}}} \left[v\right]_{\vec{e}} \overset{\text{e.g.}}{=} \begin{bmatrix} e^1_1 & e^1_2 \\ e^2_1 & e^2_2 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \qquad B = \left[E\right]_{\tilde{E}} = \left[e\right]_{\vec{\tilde{e}}} \overset{\text{e.g.}}{=} \begin{bmatrix} e^1_1 & e^1_2 \\ e^2_1 & e^2_2 \end{bmatrix}$$

$$\left[v^i\right] = \left[v\right]_{E} = \left[\tilde{E}\right]_{E} \left[v\right]_{\tilde{E}} = \left[\tilde{e}^i_j\right] \left[\tilde{v}^j\right] = \left[\tilde{e}^i_j \tilde{v}^j\right] \qquad \left[\tilde{E}\right]_{E} = \left[\tilde{e}^i_j\right]$$

$$\left[\tilde{v}^i\right] = \left[v\right]_{\tilde{E}} = \left[E\right]_{\tilde{E}} \left[v\right]_{E} = \left[e^i_j\right] \left[v^j\right] = \left[e^i_j v^j\right] \qquad \left[E\right]_{\tilde{E}} = \left[e^i_j\right]$$

$$v^i = \tilde{e}^i_j \tilde{v}^j = \sum_j \tilde{e}^i_j \tilde{v}^j = \sum_{j=1}^{\dim \mathcal{V}} \tilde{e}^i_j \tilde{v}^j = \sum_{j=1}^{n} \tilde{e}^i_j \tilde{v}^j$$

E.s.c. = Einstein summation convention

$$\tilde{v}^i = e^i_j v^j = \sum_j e^i_j v^j = \sum_{j=1}^{\dim \mathcal{V}} e^i_j v^j = \sum_{j=1}^{n} e^i_j v^j$$

E.s.c. = Einstein summation convention

change of basis

covariant basis vector	contravariant vector component scalar
$\begin{cases} \vec{\tilde{e}}_j = \vec{e}_i \tilde{e}^i_j \\ \vec{e}_j = \vec{\tilde{e}}_i e^i_j \end{cases}$	$\begin{cases} \tilde{v}^i = e^i_j v^j \\ v^i = \tilde{e}^i_j \tilde{v}^j \end{cases}$

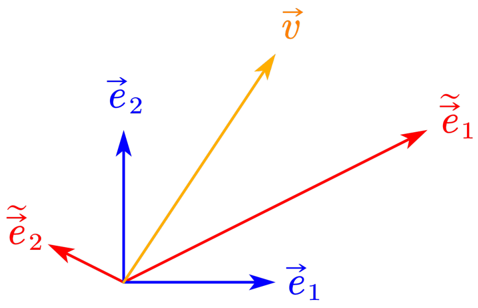

Figure 1.3: vector component

covector is basically a row vector, which is a function on column vectors

$$\overleftarrow{\alpha} : \mathcal{V} \to \mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \cdots\}$$

$$\overleftarrow{\alpha}\left(\vec{v}\right) = \overleftarrow{\alpha}\vec{v} = \begin{bmatrix}\alpha_1 & \alpha_2\end{bmatrix}\begin{bmatrix}v^1 \\ v^2\end{bmatrix} = \alpha_1 v^1 + \alpha_2 v^2 \in \mathbb{F} \overset{\text{e.g.}}{=} \mathbb{R} \qquad \forall \vec{v} \in \mathcal{V} = \mathbb{R}^2, \forall \overleftarrow{\alpha} \in \mathbb{F}^{\mathcal{V}} \overset{\text{e.g.}}{=} \mathbb{R}_2$$

linearity $\begin{cases} + & \text{additivity} \\ \cdot & \text{homogeneity} \end{cases}$

$$\begin{aligned}
\overleftarrow{\alpha}\left(\vec{u} + \vec{v}\right) &= \begin{bmatrix}\alpha_1 & \alpha_2\end{bmatrix}\begin{bmatrix}u^1 + v^1 \\ u^2 + v^2\end{bmatrix} \\
&= \alpha_1\left(u^1 + v^1\right) + \alpha_2\left(u^2 + v^2\right) \\
&= \alpha_1 u^1 + \alpha_1 v^1 + \alpha_2 u^2 + \alpha_2 v^2 && (dv) \text{ vector distributivity} \\
&= \alpha_1 u^1 + \alpha_2 u^2 + \alpha_1 v^1 + \alpha_2 v^2 && (c) \text{ commutativity} \\
&= \begin{bmatrix}\alpha_1 & \alpha_2\end{bmatrix}\begin{bmatrix}u^1 \\ u^2\end{bmatrix} + \begin{bmatrix}\alpha_1 & \alpha_2\end{bmatrix}\begin{bmatrix}v^1 \\ v^2\end{bmatrix} \\
&= \overleftarrow{\alpha}\vec{u} + \overleftarrow{\alpha}\vec{v} = \overleftarrow{\alpha}\left(\vec{u}\right) + \overleftarrow{\alpha}\left(\vec{v}\right) \\
\overleftarrow{\alpha}\left(\vec{u} + \vec{v}\right) &= \overleftarrow{\alpha}\left(\vec{u}\right) + \overleftarrow{\alpha}\left(\vec{v}\right) && \text{additivity}
\end{aligned}$$

$$\begin{aligned}
\overleftarrow{\alpha}\left(s\vec{v}\right) &= \begin{bmatrix}\alpha_1 & \alpha_2\end{bmatrix}\begin{bmatrix}sv^1 \\ sv^2\end{bmatrix} \\
&= \alpha_1\left(sv^1\right) + \alpha_2\left(sv^2\right) \\
&= \left(\alpha_1 s\right) v^1 + \left(\alpha_2 s\right) v^2 && (a) \text{ associativity} \\
&= \left(s\alpha_1\right) v^1 + \left(s\alpha_2\right) v^2 = s\left(\alpha_1 v^1 + \alpha_2 v^2\right) && (dv) \text{ vector distributivity} \\
&= \begin{bmatrix}s\alpha_1 & s\alpha_2\end{bmatrix}\begin{bmatrix}v^1 \\ v^2\end{bmatrix} = s\begin{bmatrix}\alpha_1 & \alpha_2\end{bmatrix}\begin{bmatrix}v^1 \\ v^2\end{bmatrix} \\
&= \left(s\overleftarrow{\alpha}\right)\left(\vec{v}\right) = s\overleftarrow{\alpha}\left(\vec{v}\right) \\
\overleftarrow{\alpha}\left(s\vec{v}\right) &= \left(s\overleftarrow{\alpha}\right)\left(\vec{v}\right) = s\overleftarrow{\alpha}\left(\vec{v}\right) && \text{homogeneity}
\end{aligned}$$

linearity

$$\overleftarrow{\alpha}\left(s_1\vec{u} + s_2\vec{v}\right) = s_1\overleftarrow{\alpha}\left(\vec{u}\right) + s_2\overleftarrow{\alpha}\left(\vec{v}\right) \Leftrightarrow \begin{cases} \overleftarrow{\alpha}\left(\vec{u} + \vec{v}\right) = \overleftarrow{\alpha}\left(\vec{u}\right) + \overleftarrow{\alpha}\left(\vec{v}\right) & \text{additivity} \\ \overleftarrow{\alpha}\left(s\vec{v}\right) = \left(s\overleftarrow{\alpha}\right)\left(\vec{v}\right) = s\overleftarrow{\alpha}\left(\vec{v}\right) & \text{homogeneity} \end{cases}$$

covector visualization as topographic map (geometrical meaning: contour line = isoline = isopleth = isoquant = isarithm)

$$\begin{bmatrix}2 & 1\end{bmatrix}\begin{bmatrix}x \\ y\end{bmatrix} = 2x + y$$

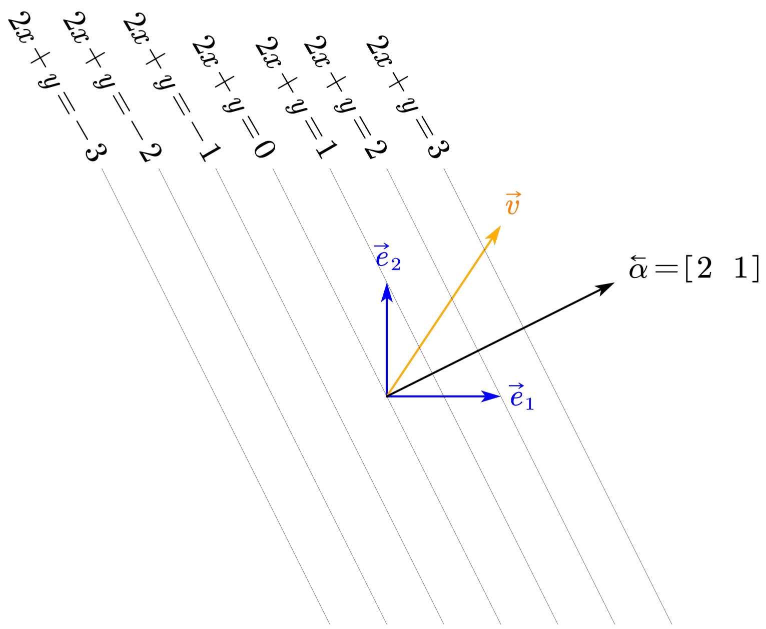

Figure 1.4: covector visualization as topographic map

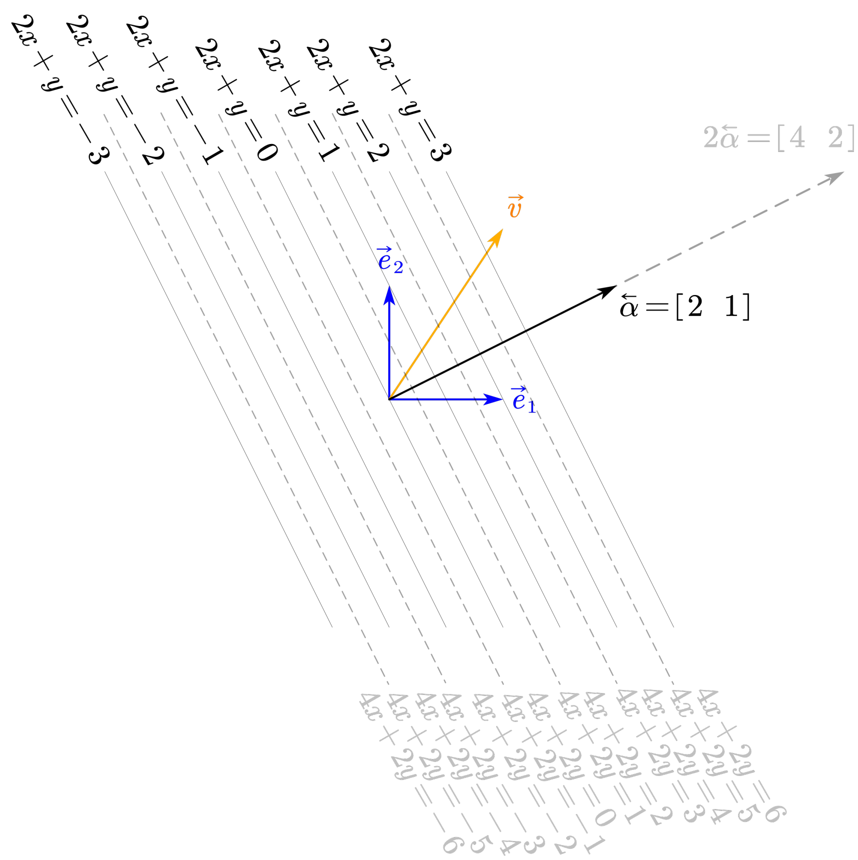

Figure 1.5: covector visualization as topographic map

$$\text{another linearity} \begin{cases} + & \text{additivity} \\ \cdot & \text{homogeneity} \end{cases}$$

$$\left(s_1\overleftarrow{\alpha} + s_2\overleftarrow{\beta}\right)\left(\vec{v}\right) = s_1\overleftarrow{\alpha}\left(\vec{v}\right) + s_2\overleftarrow{\beta}\left(\vec{v}\right) \Leftrightarrow \begin{cases} \left(\overleftarrow{\alpha} + \overleftarrow{\beta}\right)\left(\vec{v}\right) = \overleftarrow{\alpha}\left(\vec{v}\right) + \overleftarrow{\beta}\left(\vec{v}\right) & \text{additivity} \\ \left(s\overleftarrow{\alpha}\right)\left(\vec{v}\right) = s\overleftarrow{\alpha}\left(\vec{v}\right) & \text{homogeneity} \end{cases}$$

$$\left(s_1\overleftarrow{\alpha}+s_2\overleftarrow{\beta}\right)=s_1\overleftarrow{\alpha}+s_2\overleftarrow{\beta}\Leftrightarrow\begin{cases}\left(\overleftarrow{\alpha}+\overleftarrow{\beta}\right)=\overleftarrow{\alpha}+\overleftarrow{\beta} & \text{additivity}\\ \left(s\overleftarrow{\alpha}\right)=s\overleftarrow{\alpha} & \text{homogeneity}\end{cases}$$

vector space $\mathcal{V}$ = linear space

$$\vec{v}\in\mathcal{V}=\mathcal{V}\left(\mathbb{F},+,\cdot\right)=\left(\mathcal{V},\mathbb{F},+,\cdot\right)\ \text{ is a vector space over the field } \mathbb{F}:\begin{cases}+ & \text{additivity}\\ \cdot & \text{homogeneity}\end{cases}\ \text{ with element } \vec{v}$$

vector decomposition over vector space basis

$$\vec{v}=\vec{e}_1v^1+\vec{e}_2v^2=\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\begin{bmatrix}v^1\\ v^2\end{bmatrix}=\begin{bmatrix}\vec{e}\end{bmatrix}\begin{bmatrix}v^1\\ v^2\end{bmatrix}_{\vec{e}}=\begin{bmatrix}\vec{e}\end{bmatrix}\left[v\right]_{\vec{e}}=E\left[v\right]_E$$

$$\vec{v}=\vec{\tilde{e}}_1\tilde{v}^1+\vec{\tilde{e}}_2\tilde{v}^2=\begin{bmatrix}\vec{\tilde{e}}_1 & \vec{\tilde{e}}_2\end{bmatrix}\begin{bmatrix}\tilde{v}^1\\ \tilde{v}^2\end{bmatrix}=\begin{bmatrix}\vec{\tilde{e}}\end{bmatrix}\begin{bmatrix}\tilde{v}^1\\ \tilde{v}^2\end{bmatrix}_{\vec{\tilde{e}}}=\begin{bmatrix}\vec{\tilde{e}}\end{bmatrix}\left[v\right]_{\vec{\tilde{e}}}=\tilde{E}\left[v\right]_{\tilde{E}}$$

invariant vector object $\vec{v}$ but variant vector component scalar

covariant basis vector	contravariant vector component scalar
$\begin{cases}\vec{\tilde{e}}_j=\vec{e}_i\tilde{e}^i_j\\ \vec{e}_j=\vec{\tilde{e}}_ie^i_j\end{cases}$	$\begin{cases}\tilde{v}^i=e^i_jv^j\\ v^i=\tilde{e}^i_j\tilde{v}^j\end{cases}$

dual vector space $\mathcal{V}^*$ = covector space = duvector space = dual linear space

$$\overleftarrow{\alpha}\in\mathcal{V}^*=\mathcal{V}^*\left(\mathbb{F},+,\cdot\right)=\left(\mathcal{V}^*,\mathbb{F},+,\cdot\right)\ \text{ is a vector space over the field } \mathbb{F}:\begin{cases}+ & \text{additivity}\\ \cdot & \text{homogeneity}\end{cases}\ \text{ with element } \overleftarrow{\alpha}$$

$$\overleftarrow{\alpha}:\mathcal{V}\to\mathbb{F}\in\{\mathbb{R},\mathbb{C},\cdots\}$$

$$\overleftarrow{\alpha}\left(s_1\vec{u}+s_2\vec{v}\right)=s_1\overleftarrow{\alpha}\left(\vec{u}\right)+s_2\overleftarrow{\alpha}\left(\vec{v}\right)\Leftrightarrow\begin{cases}\overleftarrow{\alpha}\left(\vec{u}+\vec{v}\right)=\overleftarrow{\alpha}\left(\vec{u}\right)+\overleftarrow{\alpha}\left(\vec{v}\right) & \text{additivity}\\ \overleftarrow{\alpha}\left(s\vec{v}\right)=\left(s\overleftarrow{\alpha}\right)\left(\vec{v}\right)=s\overleftarrow{\alpha}\left(\vec{v}\right) & \text{homogeneity}\end{cases}$$

$$\left(s_1\overleftarrow{\alpha}+s_2\overleftarrow{\beta}\right)\left(\vec{v}\right)=s_1\overleftarrow{\alpha}\left(\vec{v}\right)+s_2\overleftarrow{\beta}\left(\vec{v}\right)\Leftrightarrow\begin{cases}\left(\overleftarrow{\alpha}+\overleftarrow{\beta}\right)\left(\vec{v}\right)=\overleftarrow{\alpha}\left(\vec{v}\right)+\overleftarrow{\beta}\left(\vec{v}\right) & \text{additivity}\\ \left(s\overleftarrow{\alpha}\right)\left(\vec{v}\right)=s\overleftarrow{\alpha}\left(\vec{v}\right) & \text{homogeneity}\end{cases}$$

vector space basis

$$\vec{v}\in\mathcal{V}=\operatorname{span}\left\{\vec{e}_1,\vec{e}_2\right\}=\operatorname{span}\left\{\vec{e}_i\right\}_{i=1}^{\dim\mathcal{V}=2}=\operatorname{span}\mathfrak{E}=\operatorname{span}\left\{\vec{\tilde{e}}_1,\vec{\tilde{e}}_2\right\}=\operatorname{span}\left\{\vec{\tilde{e}}_i\right\}_{i=1}^{\dim\mathcal{V}=2}=\operatorname{span}\tilde{\mathfrak{E}}$$

dual vector space basis

$$\overleftarrow{\epsilon}^1\left(\vec{e}_1\right)=\overleftarrow{\epsilon}^1\vec{e}_1=1\quad\overleftarrow{\epsilon}^1\left(\vec{e}_2\right)=\overleftarrow{\epsilon}^1\vec{e}_2=0$$
$$\overleftarrow{\epsilon}^2\left(\vec{e}_1\right)=\overleftarrow{\epsilon}^2\vec{e}_1=0\quad\overleftarrow{\epsilon}^2\left(\vec{e}_2\right)=\overleftarrow{\epsilon}^2\vec{e}_2=1$$

$$\overleftarrow{\epsilon}^i\left(\vec{e}_j\right)=\overleftarrow{\epsilon}^i\vec{e}_j=\delta^i_j=\begin{cases}1 & i=j\\ 0 & i\neq j\end{cases}$$

$$\overleftarrow{\alpha}\in\mathcal{V}^*=\operatorname{span}\left\{\overleftarrow{\epsilon}^1,\overleftarrow{\epsilon}^2\right\}=\operatorname{span}\left\{\overleftarrow{\epsilon}^i\right\}_{i=1}^{\dim\mathcal{V}^*=2}=\operatorname{span}\mathfrak{E}^*=\operatorname{span}\left\{\overleftarrow{\tilde{\epsilon}}^1,\overleftarrow{\tilde{\epsilon}}^2\right\}=\operatorname{span}\left\{\overleftarrow{\tilde{\epsilon}}^i\right\}_{i=1}^{\dim\mathcal{V}^*=2}=\operatorname{span}\tilde{\mathfrak{E}}^*$$

$$\begin{aligned}\overleftarrow{\epsilon}^1\left(\vec{v}\right)=\overleftarrow{\epsilon}^1\vec{v}&=\overleftarrow{\epsilon}^1\left(\vec{e}_1v^1+\vec{e}_2v^2\right)\\ &=v^1\overleftarrow{\epsilon}^1\left(\vec{e}_1\right)+v^2\overleftarrow{\epsilon}^1\left(\vec{e}_2\right)\\ &=v^1\cdot1+v^2\cdot0\\ &=v^1\end{aligned}$$

linearity

$$\overleftarrow{\epsilon}^1\left(\vec{e}_1\right)=\overleftarrow{\epsilon}^1\vec{e}_1=1$$
$$\overleftarrow{\epsilon}^1\left(\vec{e}_2\right)=\overleftarrow{\epsilon}^1\vec{e}_2=0$$

$$\overleftarrow{\epsilon}^1\left(\vec{v}\right)=\overleftarrow{\epsilon}^1\vec{v}=v^1$$

$$\begin{aligned}
\overleftarrow{\epsilon}^2\left(\vec{v}\right) = \overleftarrow{\epsilon}^2\vec{v} &= \overleftarrow{\epsilon}^2\left(\vec{e}_1 v^1 + \vec{e}_2 v^2\right) \\
&= v^1\overleftarrow{\epsilon}^2\left(\vec{e}_1\right) + v^2\overleftarrow{\epsilon}^2\left(\vec{e}_2\right) && \text{linearity} \\
&= v^1 \cdot 0 + v^2 \cdot 1 && \overleftarrow{\epsilon}^2\left(\vec{e}_1\right) = \overleftarrow{\epsilon}^2\vec{e}_1 = 0,\quad \overleftarrow{\epsilon}^2\left(\vec{e}_2\right) = \overleftarrow{\epsilon}^2\vec{e}_2 = 1 \\
&= v^2 \\
\overleftarrow{\epsilon}^2\left(\vec{v}\right) = \overleftarrow{\epsilon}^2\vec{v} &= v^2
\end{aligned}$$

$$\overleftarrow{\epsilon}^i\left(\vec{v}\right) = \overleftarrow{\epsilon}^i\vec{v} = v^i$$

covector decomposition over dual vector space basis

$$\begin{aligned}
\overleftarrow{\alpha}\left(\vec{v}\right) = \overleftarrow{\alpha}\vec{v} &= \overleftarrow{\alpha}\left(\vec{e}_1 v^1 + \vec{e}_2 v^2\right) \\
&= v^1\overleftarrow{\alpha}\left(\vec{e}_1\right) + v^2\overleftarrow{\alpha}\left(\vec{e}_2\right) && \text{linearity} \\
&= \overleftarrow{\epsilon}^1\left(\vec{v}\right)\overleftarrow{\alpha}\left(\vec{e}_1\right) + \overleftarrow{\epsilon}^2\left(\vec{v}\right)\overleftarrow{\alpha}\left(\vec{e}_2\right) && \begin{cases}\overleftarrow{\epsilon}^1\left(\vec{v}\right) = \overleftarrow{\epsilon}^1\vec{v} = v^1 \\ \overleftarrow{\epsilon}^2\left(\vec{v}\right) = \overleftarrow{\epsilon}^2\vec{v} = v^2\end{cases} \Leftarrow \overleftarrow{\epsilon}^i\left(\vec{v}\right) = \overleftarrow{\epsilon}^i\vec{v} = v^i \\
&= \overleftarrow{\alpha}\left(\vec{e}_1\right)\overleftarrow{\epsilon}^1\left(\vec{v}\right) + \overleftarrow{\alpha}\left(\vec{e}_2\right)\overleftarrow{\epsilon}^2\left(\vec{v}\right) && \begin{cases}\overleftarrow{\alpha}\left(\vec{e}_1\right) \in \mathbb{F} \\ \overleftarrow{\alpha}\left(\vec{e}_2\right) \in \mathbb{F}\end{cases} \\
&= \alpha_1\overleftarrow{\epsilon}^1\left(\vec{v}\right) + \alpha_2\overleftarrow{\epsilon}^2\left(\vec{v}\right) && \text{new definition or let } \begin{cases}\alpha_1 = \overleftarrow{\alpha}\left(\vec{e}_1\right) \in \mathbb{F} \\ \alpha_2 = \overleftarrow{\alpha}\left(\vec{e}_2\right) \in \mathbb{F}\end{cases} \\
&= \left(\alpha_1\overleftarrow{\epsilon}^1 + \alpha_2\overleftarrow{\epsilon}^2\right)\left(\vec{v}\right) && \text{linearity} \\
\overleftarrow{\alpha}\left(\vec{v}\right) = \overleftarrow{\alpha}\vec{v} &= \left(\alpha_1\overleftarrow{\epsilon}^1 + \alpha_2\overleftarrow{\epsilon}^2\right)\left(\vec{v}\right) \\
\overleftarrow{\alpha} &= \alpha_1\overleftarrow{\epsilon}^1 + \alpha_2\overleftarrow{\epsilon}^2 = \alpha_i\overleftarrow{\epsilon}^i && \text{E.s.c.} = \text{Einstein summation convention}
\end{aligned}$$

$$\overleftarrow{\alpha} = \alpha_1\overleftarrow{\epsilon}^1 + \alpha_2\overleftarrow{\epsilon}^2 = \alpha_i\overleftarrow{\epsilon}^i$$

$$\overleftarrow{\alpha} = \alpha_1\overleftarrow{\epsilon}^1 + \alpha_2\overleftarrow{\epsilon}^2 = \begin{bmatrix}\alpha_1 & \alpha_2\end{bmatrix}\begin{bmatrix}\overleftarrow{\epsilon}^1 \\ \overleftarrow{\epsilon}^2\end{bmatrix} = \begin{bmatrix}\alpha_1 & \alpha_2\end{bmatrix}^{\overleftarrow{\epsilon}}\begin{bmatrix}\overleftarrow{\epsilon}\end{bmatrix} = [\alpha]^{\overleftarrow{\epsilon}}\begin{bmatrix}\overleftarrow{\epsilon}\end{bmatrix} = [\alpha]^{\mathcal{E}}\mathcal{E}$$

$$\overleftarrow{\alpha} = \widetilde{\alpha}_1\widetilde{\overleftarrow{\epsilon}}^1 + \widetilde{\alpha}_2\widetilde{\overleftarrow{\epsilon}}^2 = \begin{bmatrix}\widetilde{\alpha}_1 & \widetilde{\alpha}_2\end{bmatrix}\begin{bmatrix}\widetilde{\overleftarrow{\epsilon}}^1 \\ \widetilde{\overleftarrow{\epsilon}}^2\end{bmatrix} = \begin{bmatrix}\widetilde{\alpha}_1 & \widetilde{\alpha}_2\end{bmatrix}^{\widetilde{\overleftarrow{\epsilon}}}\begin{bmatrix}\widetilde{\overleftarrow{\epsilon}}\end{bmatrix} = [\alpha]^{\widetilde{\overleftarrow{\epsilon}}}\begin{bmatrix}\widetilde{\overleftarrow{\epsilon}}\end{bmatrix} = [\alpha]^{\widetilde{\mathcal{E}}}\widetilde{\mathcal{E}}$$

$$\widetilde{\overleftarrow{\epsilon}}^i\left(\widetilde{\vec{e}}_j\right) = \overleftarrow{\epsilon}^i\left(\vec{e}_j\right) = \overleftarrow{\epsilon}^i\vec{e}_j = \delta^i_j = \begin{cases}1 & i = j \\ 0 & i \neq j\end{cases}$$

$$\begin{aligned}
\delta^i_j = \widetilde{\overleftarrow{\epsilon}}^i\left(\widetilde{\vec{e}}_j\right) &= \widetilde{\overleftarrow{\epsilon}}^i\left(\vec{e}_k\widetilde{e}^k_j\right) && \widetilde{\vec{e}}_j = \vec{e}_k\widetilde{e}^k_j \Leftarrow \widetilde{\vec{e}}_j = \vec{e}_i\widetilde{e}^i_j \\
&= \epsilon^i_h\overleftarrow{\epsilon}^h\left(\vec{e}_k\widetilde{e}^k_j\right) && \widetilde{\overleftarrow{\epsilon}}^i = \epsilon^i_h\overleftarrow{\epsilon}^h \Leftarrow \text{let } \widetilde{\overleftarrow{\epsilon}}^i = \epsilon^i_j\overleftarrow{\epsilon}^j \\
&= \epsilon^i_h\overleftarrow{\epsilon}^h\vec{e}_k\widetilde{e}^k_j && \text{linearity} \\
&= \epsilon^i_h\delta^h_k\widetilde{e}^k_j && \overleftarrow{\epsilon}^h\vec{e}_k = \delta^h_k \Leftarrow \overleftarrow{\epsilon}^i\vec{e}_j = \delta^i_j \\
&= \epsilon^i_k\widetilde{e}^k_j && \epsilon^i_h\delta^h_k = \epsilon^i_k \\
\delta^i_j &= \epsilon^i_k\widetilde{e}^k_j \\
\epsilon^i_k &= e^i_k && \delta^i_j = e^i_k\widetilde{e}^k_j \Leftrightarrow \left[\delta^i_j\right] = \left[e^i_k\widetilde{e}^k_j\right] \\
\left[\epsilon^i_j\right] = \left[e^i_j\right] &= [e]_{\widetilde{\vec{e}}} = [E]_{\widetilde{E}} = B
\end{aligned}$$

$$\Downarrow$$

$$\left[\widetilde{\epsilon}^i_j\right] = \left[\widetilde{e}^i_j\right] = \left[\widetilde{e}\right]_{\vec{e}} = \left[\widetilde{E}\right]_E = F \qquad \text{let } \overleftarrow{\epsilon}^i = \widetilde{\epsilon}^i_j\widetilde{\overleftarrow{\epsilon}}^j$$

$$\widetilde{\overleftarrow{\epsilon}}^i = \epsilon^i_j\overleftarrow{\epsilon}^j = e^i_j\overleftarrow{\epsilon}^j \Rightarrow \widetilde{\overleftarrow{\epsilon}}^i = e^i_j\overleftarrow{\epsilon}^j$$

$$\overleftarrow{\epsilon}^i = \widetilde{e}^i_j \widetilde{\overleftarrow{\epsilon}}^j = \widetilde{e}^i_j \widetilde{\overleftarrow{\epsilon}}^j \Rightarrow \overleftarrow{\epsilon}^i = \widetilde{e}^i_j \widetilde{\overleftarrow{\epsilon}}^j$$

$$\overleftarrow{\alpha} = \alpha_1 \overleftarrow{\epsilon}^1 + \alpha_2 \overleftarrow{\epsilon}^2 = \begin{bmatrix}\alpha_1 & \alpha_2\end{bmatrix} \begin{bmatrix}\overleftarrow{\epsilon}^1 \\ \overleftarrow{\epsilon}^2\end{bmatrix} = [\alpha]^{\overleftarrow{\epsilon}} \left[\overleftarrow{\epsilon}\right] = [\alpha]^{\mathcal{E}} \mathcal{E}$$

$$= [\alpha]^{\overleftarrow{\epsilon}} \left[\widetilde{e}^i_j\right] \left[\widetilde{\overleftarrow{\epsilon}}\right] = [\alpha]^{\mathcal{E}} F \widetilde{\mathcal{E}} \qquad \overleftarrow{\epsilon}^i = \widetilde{e}^i_j \widetilde{\overleftarrow{\epsilon}}^j = F^i_j \widetilde{\overleftarrow{\epsilon}}^j$$

$$= \overleftarrow{\alpha} = \widetilde{\alpha}_1 \widetilde{\overleftarrow{\epsilon}}^1 + \widetilde{\alpha}_2 \widetilde{\overleftarrow{\epsilon}}^2 = \begin{bmatrix}\widetilde{\alpha}_1 & \widetilde{\alpha}_2\end{bmatrix} \begin{bmatrix}\widetilde{\overleftarrow{\epsilon}}^1 \\ \widetilde{\overleftarrow{\epsilon}}^2\end{bmatrix} = [\alpha]^{\widetilde{\overleftarrow{\epsilon}}} \left[\widetilde{\overleftarrow{\epsilon}}\right] = [\alpha]^{\widetilde{\mathcal{E}}} \widetilde{\mathcal{E}}$$

$$= [\alpha]^{\widetilde{\overleftarrow{\epsilon}}} \left[e^i_j\right] \left[\overleftarrow{\epsilon}\right] = [\alpha]^{\widetilde{\mathcal{E}}} B \mathcal{E} \qquad \widetilde{\overleftarrow{\epsilon}}^i = e^i_j \overleftarrow{\epsilon}^j = B^i_j \overleftarrow{\epsilon}^j$$

$$[\alpha]^{\widetilde{\overleftarrow{\epsilon}}} \left[\widetilde{\overleftarrow{\epsilon}}\right] = [\alpha]^{\widetilde{\mathcal{E}}} \widetilde{\mathcal{E}} = \overleftarrow{\alpha} = [\alpha]^{\overleftarrow{\epsilon}} \left[\widetilde{e}^i_j\right] \left[\widetilde{\overleftarrow{\epsilon}}\right] = [\alpha]^{\mathcal{E}} F \widetilde{\mathcal{E}} \Rightarrow [\alpha]^{\widetilde{\overleftarrow{\epsilon}}} \left[\widetilde{\overleftarrow{\epsilon}}\right] = [\alpha]^{\widetilde{\mathcal{E}}} \widetilde{\mathcal{E}} = [\alpha]^{\overleftarrow{\epsilon}} \left[\widetilde{e}^i_j\right] \left[\widetilde{\overleftarrow{\epsilon}}\right] = [\alpha]^{\mathcal{E}} F \widetilde{\mathcal{E}}$$

$$[\alpha]^{\overleftarrow{\epsilon}} \left[\overleftarrow{\epsilon}\right] = [\alpha]^{\mathcal{E}} \mathcal{E} = \overleftarrow{\alpha} = [\alpha]^{\widetilde{\overleftarrow{\epsilon}}} \left[e^i_j\right] \left[\overleftarrow{\epsilon}\right] = [\alpha]^{\widetilde{\mathcal{E}}} B \mathcal{E} \Rightarrow [\alpha]^{\overleftarrow{\epsilon}} \left[\overleftarrow{\epsilon}\right] = [\alpha]^{\mathcal{E}} \mathcal{E} = [\alpha]^{\widetilde{\overleftarrow{\epsilon}}} \left[e^i_j\right] \left[\overleftarrow{\epsilon}\right] = [\alpha]^{\widetilde{\mathcal{E}}} B \mathcal{E}$$

$$[\alpha]^{\widetilde{\overleftarrow{\epsilon}}} \cancel{\left[\widetilde{\overleftarrow{\epsilon}}\right]} = [\alpha]^{\widetilde{\mathcal{E}}} \cancel{\widetilde{\mathcal{E}}} = [\alpha]^{\overleftarrow{\epsilon}} \left[\widetilde{e}^i_j\right] \cancel{\left[\widetilde{\overleftarrow{\epsilon}}\right]} = [\alpha]^{\mathcal{E}} F \cancel{\widetilde{\mathcal{E}}} \qquad \Rightarrow [\alpha]^{\widetilde{\overleftarrow{\epsilon}}} = [\alpha]^{\widetilde{\mathcal{E}}} = [\alpha]^{\overleftarrow{\epsilon}} \left[\widetilde{e}^i_j\right] = [\alpha]^{\mathcal{E}} F$$

$$[\alpha]^{\overleftarrow{\epsilon}} \cancel{\left[\overleftarrow{\epsilon}\right]} = [\alpha]^{\mathcal{E}} \cancel{\mathcal{E}} = [\alpha]^{\widetilde{\overleftarrow{\epsilon}}} \left[e^i_j\right] \cancel{\left[\overleftarrow{\epsilon}\right]} = [\alpha]^{\widetilde{\mathcal{E}}} B \cancel{\mathcal{E}} \qquad \Rightarrow [\alpha]^{\overleftarrow{\epsilon}} = [\alpha]^{\mathcal{E}} = [\alpha]^{\widetilde{\overleftarrow{\epsilon}}} \left[e^i_j\right] = [\alpha]^{\widetilde{\mathcal{E}}} B$$

$$\widetilde{\alpha}_j = \alpha_i \widetilde{e}^i_j \qquad [\alpha]^{\widetilde{\overleftarrow{\epsilon}}} \to \widetilde{\alpha}_j$$

$$\alpha_j = \widetilde{\alpha}_i e^i_j \qquad [\alpha]^{\overleftarrow{\epsilon}} \to \alpha_j$$

invariant covector object $\overleftarrow{\alpha}$ but variant covector component scalar

covariant basis vector	contravariant vector component scalar
$\begin{cases}\widetilde{\vec{e}}_j = \vec{e}_i \widetilde{e}^i_j \\ \vec{e}_j = \widetilde{\vec{e}}_i e^i_j\end{cases}$	$\begin{cases}\widetilde{v}^i = e^i_j v^j \\ v^i = \widetilde{e}^i_j \widetilde{v}^j\end{cases}$
covector component scalar	**dual basis covector**
$\begin{cases}\widetilde{\alpha}_j = \alpha_i \widetilde{e}^i_j \\ \alpha_j = \widetilde{\alpha}_i e^i_j\end{cases}$	$\begin{cases}\widetilde{\overleftarrow{\epsilon}}^i = e^i_j \overleftarrow{\epsilon}^j \\ \overleftarrow{\epsilon}^i = \widetilde{e}^i_j \widetilde{\overleftarrow{\epsilon}}^j\end{cases}$

$$\vec{v} = \vec{e}_1 v^1 + \vec{e}_2 v^2 = \begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix} \begin{bmatrix}v^1 \\ v^2\end{bmatrix} = \left[\vec{e}\right] [v]_{\vec{e}} = E\,[v]_E$$

$$= \begin{bmatrix}\widetilde{\vec{e}}_1 & \widetilde{\vec{e}}_2\end{bmatrix} \begin{bmatrix}e^1_1 & e^1_2 \\ e^2_1 & e^2_2\end{bmatrix} \begin{bmatrix}\widetilde{e}^1_1 & \widetilde{e}^1_2 \\ \widetilde{e}^2_1 & \widetilde{e}^2_2\end{bmatrix} \begin{bmatrix}\widetilde{v}^1 \\ \widetilde{v}^2\end{bmatrix} = \left[\widetilde{\vec{e}}\right] [e]_{\widetilde{\vec{e}}} \left[\widetilde{e}\right]_{\vec{e}} [v]_{\widetilde{\vec{e}}} = \widetilde{E}\,[E]_{\widetilde{E}} \left[\widetilde{E}\right]_E [v]_{\widetilde{E}} \qquad \begin{matrix}\vec{e}_j = \widetilde{\vec{e}}_i e^i_j \\ v^i = \widetilde{e}^i_j \widetilde{v}^j\end{matrix}$$

$$\vec{v} = \widetilde{\vec{e}}_1 \widetilde{v}^1 + \widetilde{\vec{e}}_2 \widetilde{v}^2 = \begin{bmatrix}\widetilde{\vec{e}}_1 & \widetilde{\vec{e}}_2\end{bmatrix} \begin{bmatrix}\widetilde{v}^1 \\ \widetilde{v}^2\end{bmatrix} = \left[\widetilde{\vec{e}}\right] [v]_{\widetilde{\vec{e}}} = \widetilde{E}\,[v]_{\widetilde{E}}$$

$$= \begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix} \begin{bmatrix}\widetilde{e}^1_1 & \widetilde{e}^1_2 \\ \widetilde{e}^2_1 & \widetilde{e}^2_2\end{bmatrix} \begin{bmatrix}e^1_1 & e^1_2 \\ e^2_1 & e^2_2\end{bmatrix} \begin{bmatrix}v^1 \\ v^2\end{bmatrix} = \left[\vec{e}\right] \left[\widetilde{e}\right]_{\vec{e}} [e]_{\widetilde{\vec{e}}} [v]_{\vec{e}} = E \left[\widetilde{E}\right]_E [E]_{\widetilde{E}} [v]_E \qquad \begin{matrix}\widetilde{\vec{e}}_j = \vec{e}_i \widetilde{e}^i_j \\ \widetilde{v}^i = e^i_j v^j\end{matrix}$$

$$\overleftarrow{\alpha} = \alpha_1 \overleftarrow{\epsilon}^1 + \alpha_2 \overleftarrow{\epsilon}^2 = \begin{bmatrix}\alpha_1 & \alpha_2\end{bmatrix} \begin{bmatrix}\overleftarrow{\epsilon}^1 \\ \overleftarrow{\epsilon}^2\end{bmatrix} = [\alpha]^{\overleftarrow{\epsilon}} \left[\overleftarrow{\epsilon}\right] = [\alpha]^{\mathcal{E}} \mathcal{E}$$

$$= \begin{bmatrix}\widetilde{\alpha}_1 & \widetilde{\alpha}_2\end{bmatrix} \begin{bmatrix}e^1_1 & e^1_2 \\ e^2_1 & e^2_2\end{bmatrix} \begin{bmatrix}\widetilde{e}^1_1 & \widetilde{e}^1_2 \\ \widetilde{e}^2_1 & \widetilde{e}^2_2\end{bmatrix} \begin{bmatrix}\widetilde{\overleftarrow{\epsilon}}^1 \\ \widetilde{\overleftarrow{\epsilon}}^2\end{bmatrix} = [\alpha]^{\widetilde{\overleftarrow{\epsilon}}} [e]_{\widetilde{\vec{e}}} \left[\widetilde{e}\right]_{\vec{e}} \left[\widetilde{\overleftarrow{\epsilon}}\right] = [\alpha]^{\widetilde{\mathcal{E}}} [E]_{\widetilde{E}} \left[\widetilde{E}\right]_E \widetilde{\mathcal{E}} \qquad \begin{matrix}\alpha_j = \widetilde{\alpha}_i e^i_j \\ \overleftarrow{\epsilon}^i = \widetilde{e}^i_j \widetilde{\overleftarrow{\epsilon}}^j\end{matrix}$$

$$\overleftarrow{\alpha} = \widetilde{\alpha}_1 \widetilde{\overleftarrow{\epsilon}}^1 + \widetilde{\alpha}_2 \widetilde{\overleftarrow{\epsilon}}^2 = \begin{bmatrix}\widetilde{\alpha}_1 & \widetilde{\alpha}_2\end{bmatrix} \begin{bmatrix}\widetilde{\overleftarrow{\epsilon}}^1 \\ \widetilde{\overleftarrow{\epsilon}}^2\end{bmatrix} = [\alpha]^{\widetilde{\overleftarrow{\epsilon}}} \left[\widetilde{\overleftarrow{\epsilon}}\right] = [\alpha]^{\widetilde{\mathcal{E}}} \widetilde{\mathcal{E}}$$

$$= \begin{bmatrix}\alpha_1 & \alpha_2\end{bmatrix} \begin{bmatrix}\widetilde{e}^1_1 & \widetilde{e}^1_2 \\ \widetilde{e}^2_1 & \widetilde{e}^2_2\end{bmatrix} \begin{bmatrix}e^1_1 & e^1_2 \\ e^2_1 & e^2_2\end{bmatrix} \begin{bmatrix}\overleftarrow{\epsilon}^1 \\ \overleftarrow{\epsilon}^2\end{bmatrix} = [\alpha]^{\overleftarrow{\epsilon}} \left[\widetilde{e}\right]_{\vec{e}} [e]_{\widetilde{\vec{e}}} \left[\overleftarrow{\epsilon}\right] = [\alpha]^{\mathcal{E}} \left[\widetilde{E}\right]_E [E]_{\widetilde{E}} \mathcal{E} \qquad \begin{matrix}\widetilde{\alpha}_j = \alpha_i \widetilde{e}^i_j \\ \widetilde{\overleftarrow{\epsilon}}^i = e^i_j \overleftarrow{\epsilon}^j\end{matrix}$$

https://tex.stackexchange.com/questions/654633/longtblr-error-bad-register-code-32768
seems to reach recolor upper limit, so add another \recolor{ }

$$
\begin{aligned}
\overleftarrow{\alpha}\left(\vec{v}\right) = \overleftarrow{\alpha}\vec{v} &= \left(\alpha_1\overleftarrow{\epsilon}^1+\alpha_2\overleftarrow{\epsilon}^2\right)\left(\vec{e}_1v^1+\vec{e}_2v^2\right) = \begin{bmatrix}\alpha_1 & \alpha_2\end{bmatrix}\begin{bmatrix}\overleftarrow{\epsilon}^1\\ \overleftarrow{\epsilon}^2\end{bmatrix}\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\begin{bmatrix}v^1\\ v^2\end{bmatrix}\\
&=\alpha_1\overleftarrow{\epsilon}^1\left(\vec{e}_1v^1+\vec{e}_2v^2\right)+\alpha_2\overleftarrow{\epsilon}^2\left(\vec{e}_1v^1+\vec{e}_2v^2\right) = \begin{bmatrix}\alpha_1 & \alpha_2\end{bmatrix}\begin{bmatrix}\overleftarrow{\epsilon}^1\vec{e}_1 & \overleftarrow{\epsilon}^1\vec{e}_2\\ \overleftarrow{\epsilon}^2\vec{e}_1 & \overleftarrow{\epsilon}^2\vec{e}_2\end{bmatrix}\begin{bmatrix}v^1\\ v^2\end{bmatrix}\\
&=\alpha_1\overleftarrow{\epsilon}^1\vec{e}_1v^1+\alpha_1\overleftarrow{\epsilon}^1\vec{e}_2v^2+\alpha_2\overleftarrow{\epsilon}^2\vec{e}_1v^1+\alpha_2\overleftarrow{\epsilon}^2\vec{e}_2v^2 = \begin{bmatrix}\alpha_1 & \alpha_2\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}v^1\\ v^2\end{bmatrix}\\
&=\alpha_1\cdot 1\cdot v^1+\alpha_1\cdot 0\cdot v^2+\alpha_2\cdot 0\cdot v^1+\alpha_2\cdot 1\cdot v^2 = \alpha_1v^1+\alpha_2v^2\\
= \overleftarrow{\alpha}\left(\vec{v}\right) = \overleftarrow{\alpha}\vec{v} &= \left(\tilde{\alpha}_1\tilde{\overleftarrow{\epsilon}}^1+\tilde{\alpha}_2\tilde{\overleftarrow{\epsilon}}^2\right)\left(\tilde{\vec{e}}_1\tilde{v}^1+\tilde{\vec{e}}_2\tilde{v}^2\right) = \begin{bmatrix}\tilde{\alpha}_1 & \tilde{\alpha}_2\end{bmatrix}\begin{bmatrix}\tilde{\overleftarrow{\epsilon}}^1\\ \tilde{\overleftarrow{\epsilon}}^2\end{bmatrix}\begin{bmatrix}\tilde{\vec{e}}_1 & \tilde{\vec{e}}_2\end{bmatrix}\begin{bmatrix}\tilde{v}^1\\ \tilde{v}^2\end{bmatrix}\\
&=\tilde{\alpha}_1\tilde{\overleftarrow{\epsilon}}^1\left(\tilde{\vec{e}}_1\tilde{v}^1+\tilde{\vec{e}}_2\tilde{v}^2\right)+\tilde{\alpha}_2\tilde{\overleftarrow{\epsilon}}^2\left(\tilde{\vec{e}}_1\tilde{v}^1+\tilde{\vec{e}}_2\tilde{v}^2\right) = \begin{bmatrix}\tilde{\alpha}_1 & \tilde{\alpha}_2\end{bmatrix}\begin{bmatrix}\tilde{\overleftarrow{\epsilon}}^1\tilde{\vec{e}}_1 & \tilde{\overleftarrow{\epsilon}}^1\tilde{\vec{e}}_2\\ \tilde{\overleftarrow{\epsilon}}^2\tilde{\vec{e}}_1 & \tilde{\overleftarrow{\epsilon}}^2\tilde{\vec{e}}_2\end{bmatrix}\begin{bmatrix}\tilde{v}^1\\ \tilde{v}^2\end{bmatrix}\\
&=\tilde{\alpha}_1\tilde{\overleftarrow{\epsilon}}^1\tilde{\vec{e}}_1\tilde{v}^1+\tilde{\alpha}_1\tilde{\overleftarrow{\epsilon}}^1\tilde{\vec{e}}_2\tilde{v}^2+\tilde{\alpha}_2\tilde{\overleftarrow{\epsilon}}^2\tilde{\vec{e}}_1\tilde{v}^1+\tilde{\alpha}_2\tilde{\overleftarrow{\epsilon}}^2\tilde{\vec{e}}_2\tilde{v}^2 = \begin{bmatrix}\tilde{\alpha}_1 & \tilde{\alpha}_2\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\tilde{v}^1\\ \tilde{v}^2\end{bmatrix}\\
&=\tilde{\alpha}_1\cdot 1\cdot\tilde{v}^1+\tilde{\alpha}_1\cdot 0\cdot\tilde{v}^2+\tilde{\alpha}_2\cdot 0\cdot\tilde{v}^1+\tilde{\alpha}_2\cdot 1\cdot\tilde{v}^2 = \tilde{\alpha}_1\tilde{v}^1+\tilde{\alpha}_2\tilde{v}^2
\end{aligned}
$$

$$
\begin{aligned}
\vec{v}\overleftarrow{\alpha} &= \left(\vec{e}_1v^1+\vec{e}_2v^2\right)\left(\alpha_1\overleftarrow{\epsilon}^1+\alpha_2\overleftarrow{\epsilon}^2\right) = \begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\begin{bmatrix}v^1\\ v^2\end{bmatrix}\begin{bmatrix}\alpha_1 & \alpha_2\end{bmatrix}\begin{bmatrix}\overleftarrow{\epsilon}^1\\ \overleftarrow{\epsilon}^2\end{bmatrix}\\
&=\vec{e}_1v^1\left(\alpha_1\overleftarrow{\epsilon}^1+\alpha_2\overleftarrow{\epsilon}^2\right)+\vec{e}_2v^2\left(\alpha_1\overleftarrow{\epsilon}^1+\alpha_2\overleftarrow{\epsilon}^2\right) = \begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\begin{bmatrix}v^1\alpha_1 & v^1\alpha_2\\ v^2\alpha_1 & v^2\alpha_2\end{bmatrix}\begin{bmatrix}\overleftarrow{\epsilon}^1\\ \overleftarrow{\epsilon}^2\end{bmatrix}\\
&=\vec{e}_1v^1\alpha_1\overleftarrow{\epsilon}^1+\vec{e}_1v^1\alpha_2\overleftarrow{\epsilon}^2+\vec{e}_2v^2\alpha_1\overleftarrow{\epsilon}^1+\vec{e}_2v^2\alpha_2\overleftarrow{\epsilon}^2\\
&=v^1\alpha_1\vec{e}_1\overleftarrow{\epsilon}^1+v^1\alpha_2\vec{e}_1\overleftarrow{\epsilon}^2+v^2\alpha_1\vec{e}_2\overleftarrow{\epsilon}^1+v^2\alpha_2\vec{e}_2\overleftarrow{\epsilon}^2\\
= \vec{v}\overleftarrow{\alpha} &= \left(\tilde{\vec{e}}_1\tilde{v}^1+\tilde{\vec{e}}_2\tilde{v}^2\right)\left(\tilde{\alpha}_1\tilde{\overleftarrow{\epsilon}}^1+\tilde{\alpha}_2\tilde{\overleftarrow{\epsilon}}^2\right) = \begin{bmatrix}\tilde{\vec{e}}_1 & \tilde{\vec{e}}_2\end{bmatrix}\begin{bmatrix}\tilde{v}^1\\ \tilde{v}^2\end{bmatrix}\begin{bmatrix}\tilde{\alpha}_1 & \tilde{\alpha}_2\end{bmatrix}\begin{bmatrix}\tilde{\overleftarrow{\epsilon}}^1\\ \tilde{\overleftarrow{\epsilon}}^2\end{bmatrix}\\
&=\tilde{\vec{e}}_1\tilde{v}^1\left(\tilde{\alpha}_1\tilde{\overleftarrow{\epsilon}}^1+\tilde{\alpha}_2\tilde{\overleftarrow{\epsilon}}^2\right)+\tilde{\vec{e}}_2\tilde{v}^2\left(\tilde{\alpha}_1\tilde{\overleftarrow{\epsilon}}^1+\tilde{\alpha}_2\tilde{\overleftarrow{\epsilon}}^2\right) = \begin{bmatrix}\tilde{\vec{e}}_1 & \tilde{\vec{e}}_2\end{bmatrix}\begin{bmatrix}\tilde{v}^1\tilde{\alpha}_1 & \tilde{v}^1\tilde{\alpha}_2\\ \tilde{v}^2\tilde{\alpha}_1 & \tilde{v}^2\tilde{\alpha}_2\end{bmatrix}\begin{bmatrix}\tilde{\overleftarrow{\epsilon}}^1\\ \tilde{\overleftarrow{\epsilon}}^2\end{bmatrix}\\
&=\tilde{\vec{e}}_1\tilde{v}^1\tilde{\alpha}_1\tilde{\overleftarrow{\epsilon}}^1+\tilde{\vec{e}}_1\tilde{v}^1\tilde{\alpha}_2\tilde{\overleftarrow{\epsilon}}^2+\tilde{\vec{e}}_2\tilde{v}^2\tilde{\alpha}_1\tilde{\overleftarrow{\epsilon}}^1+\tilde{\vec{e}}_2\tilde{v}^2\tilde{\alpha}_2\tilde{\overleftarrow{\epsilon}}^2\\
&=\tilde{v}^1\tilde{\alpha}_1\tilde{\vec{e}}_1\tilde{\overleftarrow{\epsilon}}^1+\tilde{v}^1\tilde{\alpha}_2\tilde{\vec{e}}_1\tilde{\overleftarrow{\epsilon}}^2+\tilde{v}^2\tilde{\alpha}_1\tilde{\vec{e}}_2\tilde{\overleftarrow{\epsilon}}^1+\tilde{v}^2\tilde{\alpha}_2\tilde{\vec{e}}_2\tilde{\overleftarrow{\epsilon}}^2
\end{aligned}
$$

$$
\begin{aligned}
\vec{v}\overleftarrow{\alpha} &= \begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\begin{bmatrix}v^1\\ v^2\end{bmatrix}\begin{bmatrix}\alpha_1 & \alpha_2\end{bmatrix}\begin{bmatrix}\overleftarrow{\epsilon}^1\\ \overleftarrow{\epsilon}^2\end{bmatrix} = \begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\begin{bmatrix}v^1\alpha_1 & v^1\alpha_2\\ v^2\alpha_1 & v^2\alpha_2\end{bmatrix}\begin{bmatrix}\overleftarrow{\epsilon}^1\\ \overleftarrow{\epsilon}^2\end{bmatrix}\\
&=\vec{e}_1v^1\alpha_1\overleftarrow{\epsilon}^1+\vec{e}_1v^1\alpha_2\overleftarrow{\epsilon}^2+\vec{e}_2v^2\alpha_1\overleftarrow{\epsilon}^1+\vec{e}_2v^2\alpha_2\overleftarrow{\epsilon}^2\\
&=v^1\alpha_1\vec{e}_1\overleftarrow{\epsilon}^1+v^1\alpha_2\vec{e}_1\overleftarrow{\epsilon}^2+v^2\alpha_1\vec{e}_2\overleftarrow{\epsilon}^1+v^2\alpha_2\vec{e}_2\overleftarrow{\epsilon}^2
\end{aligned}
$$

$$
\vec{e}_1 = \begin{bmatrix}1\\ 0\end{bmatrix}
$$

$$
=v^1\alpha_1\begin{bmatrix}1\\ 0\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix}+v^1\alpha_2\begin{bmatrix}1\\ 0\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix}+v^2\alpha_1\begin{bmatrix}0\\ 1\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix}+v^2\alpha_2\begin{bmatrix}0\\ 1\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix} \qquad \vec{e}_2 = \begin{bmatrix}0\\ 1\end{bmatrix}
$$

$$
\overleftarrow{\epsilon}^1 = \begin{bmatrix}1 & 0\end{bmatrix}
$$

$$
\overleftarrow{\epsilon}^2 = \begin{bmatrix}0 & 1\end{bmatrix}
$$

$$
=v^1\alpha_1\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}+v^1\alpha_2\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}+v^2\alpha_1\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}+v^2\alpha_2\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} \qquad = \begin{bmatrix}v^1\alpha_1 & v^1\alpha_2\\ v^2\alpha_1 & v^2\alpha_2\end{bmatrix}
$$

linear map L

$$
L:\mathcal{V}\to\mathcal{V} \Leftrightarrow \begin{cases}L:\mathcal{V}\to\mathcal{W}\\ \mathcal{W}=\mathcal{V}\end{cases}
$$

linearity

$$\overleftarrow{\alpha}\left(s_1\vec{u}+s_2\vec{v}\right)=s_1\overleftarrow{\alpha}\left(\vec{u}\right)+s_2\overleftarrow{\alpha}\left(\vec{v}\right)\Leftrightarrow\begin{cases}\overleftarrow{\alpha}\left(\vec{u}+\vec{v}\right)=\overleftarrow{\alpha}\left(\vec{u}\right)+\overleftarrow{\alpha}\left(\vec{v}\right) & \text{additivity}\\ \overleftarrow{\alpha}\left(s\vec{v}\right)=\left(s\overleftarrow{\alpha}\right)\left(\vec{v}\right)=s\overleftarrow{\alpha}\left(\vec{v}\right) & \text{homogeneity}\end{cases}$$

$$L\left(s_1\vec{u}+s_2\vec{v}\right)=s_1L\left(\vec{u}\right)+s_2L\left(\vec{v}\right)\Leftrightarrow\begin{cases}L\left(\vec{u}+\vec{v}\right)=L\left(\vec{u}\right)+L\left(\vec{v}\right) & \text{additivity}\\ L\left(s\vec{v}\right)=\left(s\,L\right)\left(\vec{v}\right)=s\,L\left(\vec{v}\right) & \text{homogeneity}\end{cases}$$

$\vec{v}$ linearly mapped to $\vec{w}$:

$$\vec{w}=L\left(\vec{v}\right)=L\left(v^1\vec{e}_1+v^2\vec{e}_2\right)=v^1L\left(\vec{e}_1\right)+v^2L\left(\vec{e}_2\right)$$

$L:\mathcal{V}\rightarrow\mathcal{V}$:

$$\begin{aligned}L\left(\vec{e}_1\right)&=L_1^1\vec{e}_1+L_1^2\vec{e}_2\\ L\left(\vec{e}_2\right)&=L_2^1\vec{e}_1+L_2^2\vec{e}_2\end{aligned}\Rightarrow L\left(\vec{e}_j\right)=\sum_{k=1}^{n}L_j^k\vec{e}_k$$

$$\begin{aligned}\vec{w}=L\left(\vec{v}\right)=L\left(v^1\vec{e}_1+v^2\vec{e}_2\right)&=v^1L\left(\vec{e}_1\right)+v^2L\left(\vec{e}_2\right)\\ &=v^1\left(L_1^1\vec{e}_1+L_1^2\vec{e}_2\right)+v^2\left(L_2^1\vec{e}_1+L_2^2\vec{e}_2\right)\\ &=\left(L_1^1v^1+L_2^1v^2\right)\vec{e}_1+\left(L_1^2v^1+L_2^2v^2\right)\vec{e}_2\\ &=w^1\vec{e}_1+w^2\vec{e}_2\end{aligned}$$

$$\begin{aligned}w^1&=L_1^1v^1+L_2^1v^2\\ w^2&=L_1^2v^1+L_2^2v^2\end{aligned}$$

$$\begin{aligned}w^1&=L_1^1v^1+L_2^1v^2\\ w^2&=L_1^2v^1+L_2^2v^2\end{aligned}\Leftrightarrow\begin{bmatrix}w^1\\w^2\end{bmatrix}=\begin{bmatrix}L_1^1 & L_2^1\\ L_1^2 & L_2^2\end{bmatrix}\begin{bmatrix}v^1\\v^2\end{bmatrix}\Rightarrow w^i=\sum_{j=1}^{n}L_j^iv^j$$

$$L=\begin{bmatrix}L_1^1 & L_2^1\\ L_1^2 & L_2^2\end{bmatrix}$$

$$\begin{aligned}L&=\begin{bmatrix}L_1^1 & L_2^1\\ L_1^2 & L_2^2\end{bmatrix}\\ &=L_1^1\begin{bmatrix}1&0\\0&0\end{bmatrix}+L_2^1\begin{bmatrix}0&1\\0&0\end{bmatrix}+L_1^2\begin{bmatrix}0&0\\1&0\end{bmatrix}+L_2^2\begin{bmatrix}0&0\\0&1\end{bmatrix}\\ &=L_1^1\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1&0\end{bmatrix}+L_2^1\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}0&1\end{bmatrix}+L_1^2\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1&0\end{bmatrix}+L_2^2\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}0&1\end{bmatrix}\\ &=\begin{bmatrix}1\\0\end{bmatrix}L_1^1\begin{bmatrix}1&0\end{bmatrix}+\begin{bmatrix}1\\0\end{bmatrix}L_2^1\begin{bmatrix}0&1\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}L_1^2\begin{bmatrix}1&0\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}L_2^2\begin{bmatrix}0&1\end{bmatrix}\end{aligned}$$

$$\vec{e}_1=\begin{bmatrix}1\\0\end{bmatrix}\quad\vec{e}_2=\begin{bmatrix}0\\1\end{bmatrix}\quad\overleftarrow{\epsilon}^1=\begin{bmatrix}1&0\end{bmatrix}\quad\overleftarrow{\epsilon}^2=\begin{bmatrix}0&1\end{bmatrix}$$

$$=L_1^1\vec{e}_1\overleftarrow{\epsilon}^1+L_2^1\vec{e}_1\overleftarrow{\epsilon}^2+L_1^2\vec{e}_2\overleftarrow{\epsilon}^1+L_2^2\vec{e}_2\overleftarrow{\epsilon}^2$$

$$\begin{aligned}&=\vec{e}_1L_1^1\overleftarrow{\epsilon}^1+\vec{e}_1L_2^1\overleftarrow{\epsilon}^2+\vec{e}_2L_1^2\overleftarrow{\epsilon}^1+\vec{e}_2L_2^2\overleftarrow{\epsilon}^2\\ &=L_j^i\vec{e}_i\overleftarrow{\epsilon}^j=\vec{e}_iL_j^i\overleftarrow{\epsilon}^j\\ L&=\vec{e}_iL_j^i\overleftarrow{\epsilon}^j\end{aligned}$$

E.s.c. = Einstein summation convention

$$\vec{v}\overleftarrow{\alpha} = \begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\begin{bmatrix}v^1\\ v^2\end{bmatrix}\begin{bmatrix}\alpha_1 & \alpha_2\end{bmatrix}\begin{bmatrix}\overleftarrow{\epsilon}^1\\ \overleftarrow{\epsilon}^2\end{bmatrix} = \begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\begin{bmatrix}v^1\alpha_1 & v^1\alpha_2\\ v^2\alpha_1 & v^2\alpha_2\end{bmatrix}\begin{bmatrix}\overleftarrow{\epsilon}^1\\ \overleftarrow{\epsilon}^2\end{bmatrix} \qquad = \begin{bmatrix}\vec{e}_i\end{bmatrix}\begin{bmatrix}v^i\alpha_j\end{bmatrix}\begin{bmatrix}\overleftarrow{\epsilon}^j\end{bmatrix}$$

$$= \vec{e}_1 v^1\alpha_1\overleftarrow{\epsilon}^1 + \vec{e}_1 v^1\alpha_2\overleftarrow{\epsilon}^2 + \vec{e}_2 v^2\alpha_1\overleftarrow{\epsilon}^1 + \vec{e}_2 v^2\alpha_2\overleftarrow{\epsilon}^2 \qquad = \vec{e}_i v^i\alpha_j\overleftarrow{\epsilon}^j$$

$$\vec{e}_1 = \begin{bmatrix}1\\0\end{bmatrix}$$

$$= v^1\alpha_1\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix} + v^1\alpha_2\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix} + v^2\alpha_1\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix} + v^2\alpha_2\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix} \qquad \vec{e}_2 = \begin{bmatrix}0\\1\end{bmatrix}$$

$$\overleftarrow{\epsilon}^1 = \begin{bmatrix}1 & 0\end{bmatrix}$$

$$\overleftarrow{\epsilon}^2 = \begin{bmatrix}0 & 1\end{bmatrix}$$

$$= v^1\alpha_1\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} + v^1\alpha_2\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} + v^2\alpha_1\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix} + v^2\alpha_2\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} \qquad = \begin{bmatrix}v^1\alpha_1 & v^1\alpha_2\\ v^2\alpha_1 & v^2\alpha_2\end{bmatrix}$$

$$\vec{v}\overleftarrow{\alpha} = \left(\vec{e}_1 v^1 + \vec{e}_2 v^2\right)\left(\alpha_1\overleftarrow{\epsilon}^1 + \alpha_2\overleftarrow{\epsilon}^2\right) = \begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\begin{bmatrix}v^1\\ v^2\end{bmatrix}\begin{bmatrix}\alpha_1 & \alpha_2\end{bmatrix}\begin{bmatrix}\overleftarrow{\epsilon}^1\\ \overleftarrow{\epsilon}^2\end{bmatrix}$$

$$= \vec{e}_1 v^1\left(\alpha_1\overleftarrow{\epsilon}^1 + \alpha_2\overleftarrow{\epsilon}^2\right) + \vec{e}_2 v^2\left(\alpha_1\overleftarrow{\epsilon}^1 + \alpha_2\overleftarrow{\epsilon}^2\right) = \begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\begin{bmatrix}v^1\alpha_1 & v^1\alpha_2\\ v^2\alpha_1 & v^2\alpha_2\end{bmatrix}\begin{bmatrix}\overleftarrow{\epsilon}^1\\ \overleftarrow{\epsilon}^2\end{bmatrix} \qquad = \begin{bmatrix}\vec{e}_i\end{bmatrix}\begin{bmatrix}v^i\alpha_j\end{bmatrix}\begin{bmatrix}\overleftarrow{\epsilon}^j\end{bmatrix}$$

$$= \vec{e}_1 v^1\alpha_1\overleftarrow{\epsilon}^1 + \vec{e}_1 v^1\alpha_2\overleftarrow{\epsilon}^2 + \vec{e}_2 v^2\alpha_1\overleftarrow{\epsilon}^1 + \vec{e}_2 v^2\alpha_2\overleftarrow{\epsilon}^2 \qquad = \vec{e}_i v^i\alpha_j\overleftarrow{\epsilon}^j$$

$$= \vec{v}\overleftarrow{\alpha} = \left(\widetilde{\vec{e}}_1\widetilde{v}^1 + \widetilde{\vec{e}}_2\widetilde{v}^2\right)\left(\widetilde{\alpha}_1\widetilde{\overleftarrow{\epsilon}}^1 + \widetilde{\alpha}_2\widetilde{\overleftarrow{\epsilon}}^2\right) = \begin{bmatrix}\widetilde{\vec{e}}_1 & \widetilde{\vec{e}}_2\end{bmatrix}\begin{bmatrix}\widetilde{v}^1\\ \widetilde{v}^2\end{bmatrix}\begin{bmatrix}\widetilde{\alpha}_1 & \widetilde{\alpha}_2\end{bmatrix}\begin{bmatrix}\widetilde{\overleftarrow{\epsilon}}^1\\ \widetilde{\overleftarrow{\epsilon}}^2\end{bmatrix}$$

$$= \widetilde{\vec{e}}_1\widetilde{v}^1\left(\widetilde{\alpha}_1\widetilde{\overleftarrow{\epsilon}}^1 + \widetilde{\alpha}_2\widetilde{\overleftarrow{\epsilon}}^2\right) + \widetilde{\vec{e}}_2\widetilde{v}^2\left(\widetilde{\alpha}_1\widetilde{\overleftarrow{\epsilon}}^1 + \widetilde{\alpha}_2\widetilde{\overleftarrow{\epsilon}}^2\right) = \begin{bmatrix}\widetilde{\vec{e}}_1 & \widetilde{\vec{e}}_2\end{bmatrix}\begin{bmatrix}\widetilde{v}^1\widetilde{\alpha}_1 & \widetilde{v}^1\widetilde{\alpha}_2\\ \widetilde{v}^2\widetilde{\alpha}_1 & \widetilde{v}^2\widetilde{\alpha}_2\end{bmatrix}\begin{bmatrix}\widetilde{\overleftarrow{\epsilon}}^1\\ \widetilde{\overleftarrow{\epsilon}}^2\end{bmatrix} \qquad = \begin{bmatrix}\widetilde{\vec{e}}_i\end{bmatrix}\begin{bmatrix}\widetilde{v}^i\widetilde{\alpha}_j\end{bmatrix}\begin{bmatrix}\widetilde{\overleftarrow{\epsilon}}^j\end{bmatrix}$$

$$= \widetilde{\vec{e}}_1\widetilde{v}^1\widetilde{\alpha}_1\widetilde{\overleftarrow{\epsilon}}^1 + \widetilde{\vec{e}}_1\widetilde{v}^1\widetilde{\alpha}_2\widetilde{\overleftarrow{\epsilon}}^2 + \widetilde{\vec{e}}_2\widetilde{v}^2\widetilde{\alpha}_1\widetilde{\overleftarrow{\epsilon}}^1 + \widetilde{\vec{e}}_2\widetilde{v}^2\widetilde{\alpha}_2\widetilde{\overleftarrow{\epsilon}}^2 \qquad = \widetilde{\vec{e}}_i\widetilde{v}^i\widetilde{\alpha}_j\widetilde{\overleftarrow{\epsilon}}^j$$

$$\vec{e}_i = \widetilde{\vec{e}}_h e^h_i \Leftarrow \vec{e}_j = \widetilde{\vec{e}}_i e^i_j$$

$$L = \vec{e}_i L^i_j \overleftarrow{\epsilon}^j = \widetilde{\vec{e}}_h e^h_i L^i_j \widetilde{e}^j_k \widetilde{\overleftarrow{\epsilon}}^k$$

$$\overleftarrow{\epsilon}^j = \widetilde{e}^j_k \widetilde{\overleftarrow{\epsilon}}^k \Leftarrow \overleftarrow{\epsilon}^i = \widetilde{e}^i_j \widetilde{\overleftarrow{\epsilon}}^j$$

$$\widetilde{\vec{e}}_i = \vec{e}_h \widetilde{e}^h_i \Leftarrow \widetilde{\vec{e}}_j = \vec{e}_i \widetilde{e}^i_j$$

$$= L = \widetilde{\vec{e}}_i \widetilde{L}^i_j \widetilde{\overleftarrow{\epsilon}}^j = \vec{e}_h \widetilde{e}^h_i \widetilde{L}^i_j e^j_k \overleftarrow{\epsilon}^k$$

$$\widetilde{\overleftarrow{\epsilon}}^j = e^j_k \overleftarrow{\epsilon}^k \Leftarrow \widetilde{\overleftarrow{\epsilon}}^i = e^i_j \overleftarrow{\epsilon}^j$$

$$\widetilde{\vec{e}}_i \widetilde{L}^i_j \widetilde{\overleftarrow{\epsilon}}^j = L = \widetilde{\vec{e}}_h e^h_i L^i_j \widetilde{e}^j_k \widetilde{\overleftarrow{\epsilon}}^k \qquad \Rightarrow \widetilde{\vec{e}}_i \widetilde{L}^i_j \widetilde{\overleftarrow{\epsilon}}^j = \widetilde{\vec{e}}_h e^h_i L^i_j \widetilde{e}^j_k \widetilde{\overleftarrow{\epsilon}}^k$$

$$\vec{e}_i L^i_j \overleftarrow{\epsilon}^j = L = \vec{e}_h \widetilde{e}^h_i \widetilde{L}^i_j e^j_k \overleftarrow{\epsilon}^k \qquad \Rightarrow \vec{e}_i L^i_j \overleftarrow{\epsilon}^j = \vec{e}_h \widetilde{e}^h_i \widetilde{L}^i_j e^j_k \overleftarrow{\epsilon}^k$$

$$\widetilde{\vec{e}}_i \widetilde{L}^i_j \widetilde{\overleftarrow{\epsilon}}^j = \widetilde{\vec{e}}_h e^h_i L^i_j \widetilde{e}^j_k \widetilde{\overleftarrow{\epsilon}}^k = \widetilde{\vec{e}}_i e^i_h L^h_k \widetilde{e}^k_j \widetilde{\overleftarrow{\epsilon}}^j \qquad \text{RHS}: \begin{matrix} h \to i \\ i \to h \\ j \to k \\ k \to j \end{matrix}$$

$$\vec{e}_i L^i_j \overleftarrow{\epsilon}^j = \vec{e}_h \widetilde{e}^h_i \widetilde{L}^i_j e^j_k \overleftarrow{\epsilon}^k = \vec{e}_i \widetilde{e}^i_h \widetilde{L}^h_k e^k_j \overleftarrow{\epsilon}^j \qquad \text{RHS}: \begin{matrix} h \to i \\ i \to h \\ j \to k \\ k \to j \end{matrix}$$

$$\cancel{\widetilde{\vec{e}}_i} \widetilde{L}^i_j \cancel{\widetilde{\overleftarrow{\epsilon}}^j} = \cancel{\widetilde{\vec{e}}_i} e^i_h L^h_k \widetilde{e}^k_j \cancel{\widetilde{\overleftarrow{\epsilon}}^j} \qquad \Rightarrow \widetilde{L}^i_j = e^i_h L^h_k \widetilde{e}^k_j$$

$$\cancel{\vec{e}_i} L^i_j \cancel{\overleftarrow{\epsilon}^j} = \cancel{\vec{e}_i} \widetilde{e}^i_h \widetilde{L}^h_k e^k_j \cancel{\overleftarrow{\epsilon}^j} \qquad \Rightarrow L^i_j = \widetilde{e}^i_h \widetilde{L}^h_k e^k_j$$

$(0,1)$-tensor
covariant
basis vector

$$\begin{cases}\tilde{\vec e}_j = \vec e_i \tilde e^i_j \\ \vec e_j = \tilde{\vec e}_i e^i_j\end{cases}$$

covector component scalar

$$\begin{cases}\tilde\alpha_j = \alpha_i \tilde e^i_j \\ \alpha_j = \tilde\alpha_i e^i_j\end{cases}$$

$(1,0)$-tensor
contravariant
vector component scalar

$$\begin{cases}\tilde v^i = e^i_j v^j \\ v^i = \tilde e^i_j \tilde v^j\end{cases}$$

dual basis covector

$$\begin{cases}\tilde{\overleftarrow\epsilon}^i = e^i_j \overleftarrow\epsilon^j \\ \overleftarrow\epsilon^i = \tilde e^i_j \tilde{\overleftarrow\epsilon}^j\end{cases}$$

$(1,1)$-tensor

linear map

$$\begin{cases}\tilde L^i_j = e^i_h L^h_k \tilde e^k_j \\ L^i_j = \tilde e^i_h \tilde L^h_k e^k_j\end{cases}$$

$$\vec v = \vec e_1 v^1 + \vec e_2 v^2 = \begin{bmatrix}\vec e_1 & \vec e_2\end{bmatrix}\begin{bmatrix}v^1\\v^2\end{bmatrix} = \begin{bmatrix}\vec e\end{bmatrix}\begin{bmatrix}v^1\\v^2\end{bmatrix}_{\vec e} = \begin{bmatrix}\vec e\end{bmatrix}\begin{bmatrix}v\end{bmatrix}_{\vec e} = E\begin{bmatrix}v\end{bmatrix}_E$$

$$\vec v = \tilde{\vec e}_1 \tilde v^1 + \tilde{\vec e}_2 \tilde v^2 = \begin{bmatrix}\tilde{\vec e}_1 & \tilde{\vec e}_2\end{bmatrix}\begin{bmatrix}\tilde v^1\\\tilde v^2\end{bmatrix} = \begin{bmatrix}\tilde{\vec e}\end{bmatrix}\begin{bmatrix}\tilde v^1\\\tilde v^2\end{bmatrix}_{\tilde{\vec e}} = \begin{bmatrix}\tilde{\vec e}\end{bmatrix}\begin{bmatrix}v\end{bmatrix}_{\tilde{\vec e}} = \tilde E\begin{bmatrix}v\end{bmatrix}_{\tilde E}$$

$$\overleftarrow\alpha = \alpha_1 \overleftarrow\epsilon^1 + \alpha_2 \overleftarrow\epsilon^2 = \begin{bmatrix}\alpha_1 & \alpha_2\end{bmatrix}\begin{bmatrix}\overleftarrow\epsilon^1\\\overleftarrow\epsilon^2\end{bmatrix} = \begin{bmatrix}\alpha_1 & \alpha_2\end{bmatrix}^{\overleftarrow\epsilon}\begin{bmatrix}\overleftarrow\epsilon\end{bmatrix} = \begin{bmatrix}\alpha\end{bmatrix}^{\overleftarrow\epsilon}\begin{bmatrix}\overleftarrow\epsilon\end{bmatrix} = \begin{bmatrix}\alpha\end{bmatrix}^{\mathcal E}\mathcal E$$

$$\overleftarrow\alpha = \tilde\alpha_1 \tilde{\overleftarrow\epsilon}^1 + \tilde\alpha_2 \tilde{\overleftarrow\epsilon}^2 = \begin{bmatrix}\tilde\alpha_1 & \tilde\alpha_2\end{bmatrix}\begin{bmatrix}\tilde{\overleftarrow\epsilon}^1\\\tilde{\overleftarrow\epsilon}^2\end{bmatrix} = \begin{bmatrix}\tilde\alpha_1 & \tilde\alpha_2\end{bmatrix}^{\tilde{\overleftarrow\epsilon}}\begin{bmatrix}\tilde{\overleftarrow\epsilon}\end{bmatrix} = \begin{bmatrix}\alpha\end{bmatrix}^{\tilde{\overleftarrow\epsilon}}\begin{bmatrix}\tilde{\overleftarrow\epsilon}\end{bmatrix} = \begin{bmatrix}\alpha\end{bmatrix}^{\tilde{\mathcal E}}\tilde{\mathcal E}$$

$$\mathcal V^* \ni \overset{\leftarrow}{\vec e} = \begin{cases}\begin{bmatrix}\vec e_1 & \vec e_2\end{bmatrix} = \begin{bmatrix}\vec e\end{bmatrix} = E \\ \begin{bmatrix}\tilde{\vec e}_1 & \tilde{\vec e}_2\end{bmatrix} = \begin{bmatrix}\tilde{\vec e}\end{bmatrix} = \tilde E\end{cases}$$

$$\mathcal V \ni \overset{\rightarrow}{\overleftrightarrow v} = \begin{cases}\begin{bmatrix}v^1\\v^2\end{bmatrix} = \begin{bmatrix}v^1\\v^2\end{bmatrix}_{\vec e} = \begin{bmatrix}v\end{bmatrix}_{\vec e} = \begin{bmatrix}v\end{bmatrix}_E \\ \begin{bmatrix}\tilde v^1\\\tilde v^2\end{bmatrix} = \begin{bmatrix}\tilde v^1\\\tilde v^2\end{bmatrix}_{\tilde{\vec e}} = \begin{bmatrix}v\end{bmatrix}_{\tilde{\vec e}} = \begin{bmatrix}v\end{bmatrix}_{\tilde E}\end{cases}$$

$$\mathcal V^* \ni \overset{\leftarrow}{\overleftrightarrow\alpha} = \begin{cases}\begin{bmatrix}\alpha_1 & \alpha_2\end{bmatrix} = \begin{bmatrix}\alpha_1 & \alpha_2\end{bmatrix}^{\overleftarrow\epsilon} = \begin{bmatrix}\alpha\end{bmatrix}^{\overleftarrow\epsilon} = \begin{bmatrix}\alpha\end{bmatrix}^{\mathcal E} \\ \begin{bmatrix}\tilde\alpha_1 & \tilde\alpha_2\end{bmatrix} = \begin{bmatrix}\tilde\alpha_1 & \tilde\alpha_2\end{bmatrix}^{\tilde{\overleftarrow\epsilon}} = \begin{bmatrix}\alpha\end{bmatrix}^{\tilde{\overleftarrow\epsilon}} = \begin{bmatrix}\alpha\end{bmatrix}^{\tilde{\mathcal E}}\end{cases}$$

$$\mathcal V \ni \overset{\rightarrow}{\overleftarrow\epsilon} = \begin{cases}\begin{bmatrix}\overleftarrow\epsilon^1\\\overleftarrow\epsilon^2\end{bmatrix} = \begin{bmatrix}\overleftarrow\epsilon\end{bmatrix} = \mathcal E \\ \begin{bmatrix}\tilde{\overleftarrow\epsilon}^1\\\tilde{\overleftarrow\epsilon}^2\end{bmatrix} = \begin{bmatrix}\tilde{\overleftarrow\epsilon}\end{bmatrix} = \tilde{\mathcal E}\end{cases}$$

$$\mathcal V \otimes \mathcal V^* \ni \overset{\rightarrow\leftarrow}{L} = L = \begin{cases}\begin{bmatrix}1\\0\end{bmatrix} L^1_1 \begin{bmatrix}1 & 0\end{bmatrix} + \begin{bmatrix}1\\0\end{bmatrix} L^1_2 \begin{bmatrix}0 & 1\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix} L^2_1 \begin{bmatrix}1 & 0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix} L^2_2 \begin{bmatrix}0 & 1\end{bmatrix} = \vec e_i L^i_j \overleftarrow\epsilon^j \\ \begin{bmatrix}1\\0\end{bmatrix} \tilde L^1_1 \begin{bmatrix}1 & 0\end{bmatrix} + \begin{bmatrix}1\\0\end{bmatrix} \tilde L^1_2 \begin{bmatrix}0 & 1\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix} \tilde L^2_1 \begin{bmatrix}1 & 0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix} \tilde L^2_2 \begin{bmatrix}0 & 1\end{bmatrix} = \tilde{\vec e}_i \tilde L^i_j \tilde{\overleftarrow\epsilon}^j\end{cases}$$

$$\vec v = \cancel{\overset{\leftarrow\ \rightarrow}{\vec e\,\overleftrightarrow v}} = \overset{\leftarrow\ \rightarrow}{\vec e\,\overleftrightarrow v} = \begin{bmatrix}\vec e_1 & \vec e_2\end{bmatrix}\begin{bmatrix}v^1\\v^2\end{bmatrix} = \begin{bmatrix}\vec e_1 & \vec e_2\end{bmatrix}\begin{bmatrix}\overleftarrow\epsilon^1\\\overleftarrow\epsilon^2\end{bmatrix}\begin{bmatrix}\vec e_1 & \vec e_2\end{bmatrix}\begin{bmatrix}v^1\\v^2\end{bmatrix}$$

$$\overleftarrow\alpha = \cancel{\overset{\leftarrow\ \rightarrow}{\overleftrightarrow\alpha\,\overleftarrow\epsilon}} = \overset{\leftarrow\ \rightarrow}{\overleftrightarrow\alpha\,\overleftarrow\epsilon} = \begin{bmatrix}\alpha_1 & \alpha_2\end{bmatrix}\begin{bmatrix}\overleftarrow\epsilon^1\\\overleftarrow\epsilon^2\end{bmatrix} = \begin{bmatrix}\alpha_1 & \alpha_2\end{bmatrix}\begin{bmatrix}\overleftarrow\epsilon^1\\\overleftarrow\epsilon^2\end{bmatrix}\begin{bmatrix}\vec e_1 & \vec e_2\end{bmatrix}\begin{bmatrix}\overleftarrow\epsilon^1\\\overleftarrow\epsilon^2\end{bmatrix}$$

$$\begin{aligned}\begin{bmatrix}\vec e_1 & \vec e_2\end{bmatrix}\begin{bmatrix}w^1\\w^2\end{bmatrix} = \vec w = \cancel{\overset{\rightarrow\leftarrow}{L}\,\vec v} = \overset{\rightarrow\leftarrow}{L}\,\vec v = \overset{\rightarrow\leftarrow}{L}\,\vec v = L\vec v = \vec e_i L^i_j \overleftarrow\epsilon^j \vec e_k v^k \\ = \begin{bmatrix}\vec e_1 & \vec e_2\end{bmatrix}\begin{bmatrix}L^1_1 & L^1_2\\L^2_1 & L^2_2\end{bmatrix}\begin{bmatrix}v^1\\v^2\end{bmatrix} = \begin{bmatrix}\vec e_1 & \vec e_2\end{bmatrix}\begin{bmatrix}L^1_1 & L^1_2\\L^2_1 & L^2_2\end{bmatrix}\begin{bmatrix}\overleftarrow\epsilon^1\\\overleftarrow\epsilon^2\end{bmatrix}\begin{bmatrix}\vec e_1 & \vec e_2\end{bmatrix}\begin{bmatrix}v^1\\v^2\end{bmatrix} \\ = \begin{bmatrix}\vec e_1 & \vec e_2\end{bmatrix}\begin{bmatrix}L^1_1 v^1 + L^1_2 v^2\\L^2_1 v^1 + L^2_2 v^2\end{bmatrix}\end{aligned}$$

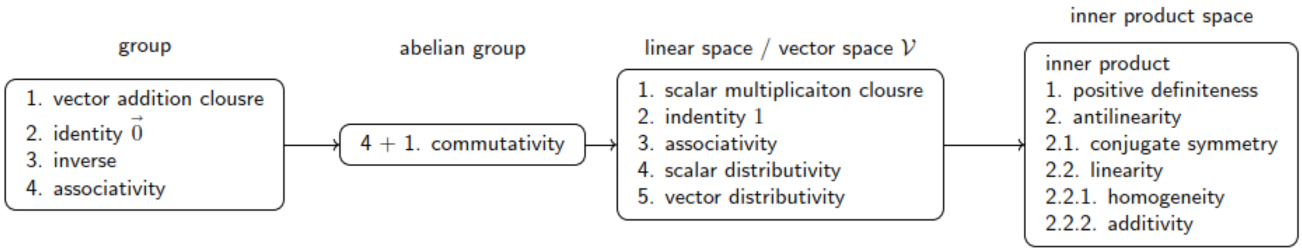

Figure 1.6: inner product space axiom construction

in physics,

inner product	metric / norm	
$\langle\cdot\vert\cdot\rangle = \langle\Box\vert\Box\rangle : \mathcal{V}^2 \to \mathbb{R}$	$\eta = \eta(\Box,\Box) : \mathcal{V}^2 \to \mathbb{F} \in \{\mathbb{C},\mathbb{R},\cdots\}$	function (f)
$\left\langle\vec{v}\vert\vec{v}\right\rangle \geq 0 \quad ,\left\langle\vec{v}\vert\vec{v}\right\rangle = 0 \Leftrightarrow \vec{v} = \vec{0}$	$\left\Vert\vec{v}\right\Vert^2 = \eta\left(\vec{v},\vec{v}\right) \gtreqless 0$	positivity (p)
$\left\langle\vec{u}\vert\vec{v}\right\rangle = \left\langle\vec{v}\vert\vec{u}\right\rangle^*$	$\eta\left(\vec{u},\vec{v}\right) = \eta\left(\vec{v},\vec{u}\right)$	symmetry(s)
$\left\langle\vec{u}\vert s\vec{v}\right\rangle = s\left\langle\vec{u}\vert\vec{v}\right\rangle$	$s\eta\left(\vec{u},\vec{v}\right) = \eta\left(s\vec{u},\vec{v}\right) = \eta\left(\vec{u},s\vec{v}\right)$	homogeneity (h)
$\left\langle\vec{u}\vert\vec{v}+\vec{w}\right\rangle = \left\langle\vec{u}\vert\vec{v}\right\rangle + \left\langle\vec{u}\vert\vec{w}\right\rangle$	$\left\Vert\vec{u}+\vec{v}\right\Vert \gtreqless \left\Vert\vec{u}\right\Vert + \left\Vert\vec{v}\right\Vert$	additivity (a)

mathematically,

inner product	norm	metric / distance function	
$\langle\cdot\vert\cdot\rangle = \langle\Box\vert\Box\rangle : \mathcal{V}^2 \xrightarrow{g} \mathbb{F}$	$\Vert\cdot\Vert = \Vert\Box\Vert : \mathcal{V} \to \mathbb{R}$	$d(\cdot,\cdot) = d(\Box,\Box) : X^2 \to \mathbb{R}$	function (f)
$\left\langle\vec{v}\vert\vec{v}\right\rangle \geq 0 \quad ,\left\langle\vec{v}\vert\vec{v}\right\rangle = 0 \Leftrightarrow \vec{v} = \vec{0}$	$\left\Vert\vec{v}\right\Vert \geq 0, \quad \left\Vert\vec{v}\right\Vert = 0 \Leftrightarrow \vec{v} = \vec{0}$	$d(x,y) \geq 0, \quad d(x,y) = 0 \Leftrightarrow x = y$	positivity (p)
$\left\langle\vec{u}\vert\vec{v}\right\rangle = \left\langle\vec{v}\vert\vec{u}\right\rangle^*$		$d(x,y) = d(y,x)$	symmetry(s)
$\left\langle\vec{u}\vert s\vec{v}\right\rangle = s\left\langle\vec{u}\vert\vec{v}\right\rangle$	$\left\Vert s\vec{v}\right\Vert = \vert s\vert\left\Vert\vec{v}\right\Vert$		homogeneity (h)
$\left\langle\vec{u}\vert\vec{v}+\vec{w}\right\rangle = \left\langle\vec{u}\vert\vec{v}\right\rangle + \left\langle\vec{u}\vert\vec{w}\right\rangle$	$\left\Vert\vec{u}+\vec{v}\right\Vert \leq \left\Vert\vec{u}\right\Vert + \left\Vert\vec{v}\right\Vert$	$d(x,y) \leq d(x,z) + d(z,y)$	additivity (a)

[1]inner product

$$\langle\Box\vert\Box\rangle : \mathcal{V}\times\mathcal{V} \to \mathbb{F}$$

$$g(\Box,\Box) = \langle\Box\vert\Box\rangle : \mathcal{V}\times\mathcal{V} \to \mathbb{F}$$

$$\langle\Box\vert\Box\rangle : \mathcal{V}^2 \xrightarrow{g} \mathbb{F}$$

$\langle\Box\vert\Box\rangle$ is a (complete) inner product on the vector space $\mathcal{V}$ (not necessarily complete, use $\langle\Box,\Box\rangle$)

$\Updownarrow$

$$\begin{cases} \langle\Box\vert\Box\rangle : \mathcal{V}\times\mathcal{V} \to \mathbb{F} \qquad \mathcal{V} = \mathcal{V}(\mathbb{F},+,\cdot) = (\mathcal{V},\mathbb{F},+,\cdot) & \mathcal{V} \text{ is a vector space over the field } \mathbb{F} \\ \left\langle\vec{v}\vert\vec{v}\right\rangle \geq 0, \quad \left\langle\vec{v}\vert\vec{v}\right\rangle = 0 \Leftrightarrow \vec{v} = \vec{0} \qquad \forall\vec{v}\in\mathcal{V} & (pd) \text{ positive definiteness} \\ \left\langle s\vec{u}\vert t\left(\vec{v}+\vec{w}\right)\right\rangle = s^*t\left(\left\langle\vec{u}\vert\vec{v}\right\rangle + \left\langle\vec{u}\vert\vec{w}\right\rangle\right) \qquad \forall s,t\in\mathbb{F}, \forall\vec{u},\vec{v},\vec{w}\in\mathcal{V} & (al) \text{ antilinearity} \end{cases}$$

$\Updownarrow$

$$\begin{cases} \langle\Box\vert\Box\rangle : \mathcal{V}\times\mathcal{V} \to \mathbb{F} \qquad \mathcal{V} = \mathcal{V}(\mathbb{F},+,\cdot) = (\mathcal{V},\mathbb{F},+,\cdot) & \mathcal{V} \text{ is a vector space over the field } \mathbb{F} \\ \left\langle\vec{v}\vert\vec{v}\right\rangle \geq 0, \quad \left\langle\vec{v}\vert\vec{v}\right\rangle = 0 \Leftrightarrow \vec{v} = \vec{0} \qquad \forall\vec{v}\in\mathcal{V} & (pd) \text{ positive definiteness} \\ \left\langle\vec{u}\vert\vec{v}\right\rangle = \left\langle\vec{v}\vert\vec{u}\right\rangle^* \qquad \forall\vec{u},\vec{v}\in\mathcal{V} & (cs) \text{ conjugate symmetry} \\ \begin{cases} \left\langle\vec{u}\vert s\vec{v}\right\rangle = s\left\langle\vec{u}\vert\vec{v}\right\rangle \qquad \forall s\in\mathbb{F}, \forall\vec{u},\vec{v}\in\mathcal{V} & (h) \text{ homogeneity} \\ \left\langle\vec{u}\vert\vec{v}+\vec{w}\right\rangle = \left\langle\vec{u}\vert\vec{v}\right\rangle + \left\langle\vec{u}\vert\vec{w}\right\rangle \qquad \forall\vec{u},\vec{v},\vec{w}\in\mathcal{V} & (a) \text{ additivity} \end{cases} & (l) \text{ linearity} \end{cases}$$

norm

$$\Vert\cdot\Vert : \mathcal{V} \to \mathbb{R}$$

[1]metric/distance: positive definiteness $d(x,y) \geq 0$ 一定有嗎? 一個顯然的例子就是, 在相對論時空下, 我們取了洛倫茲度規, 此時的向量就會根據其平方值大於零, 等於零, 小於零三種情況分爲類時, 類光, 類空三種向量(洛倫茲度規也分兩種等價的號差, 所以這仨向量名稱的用法並不統一).

$\|\square\|$ is a norm on the vector space $\mathcal{V}$

$\Updownarrow$

$$\begin{cases} \|\square\| : \mathcal{V} \to \mathbb{R} \qquad \mathcal{V} = \mathcal{V}(\mathbb{F}, +, \cdot) = (\mathcal{V}, \mathbb{F}, +, \cdot) & \mathcal{V} \text{ is a vector space over the field } \mathbb{F} \\ \left\|\vec{v}\right\| \geq 0, \quad \left\|\vec{v}\right\| = 0 \Leftrightarrow \vec{v} = \vec{0} \qquad \forall \vec{v} \in \mathcal{V} & (pd) \text{ positive definiteness} \\ \left\|s\vec{v}\right\| = |s| \left\|\vec{v}\right\| \qquad \forall s \in \mathbb{F}, \forall \vec{v} \in \mathcal{V} & (ah) \text{ absolute homogeneity} \\ \left\|\vec{u} + \vec{v}\right\| \leq \left\|\vec{u}\right\| + \left\|\vec{v}\right\| \qquad \forall \vec{u}, \vec{v} \in \mathcal{V} & (ti) \text{ subadditivity/triangle inequality} \end{cases}$$

$$\begin{aligned} \left\|\vec{v}\right\| &= \sqrt{\vec{v} \cdot \vec{v}} = +\sqrt{\vec{v} \cdot \vec{v}} = \left(\vec{v} \cdot \vec{v}\right)^{\frac{1}{2}} \\ &= \sqrt{\left(\vec{v}, \vec{v}\right)} = \sqrt{\left\langle \vec{v}, \vec{v} \right\rangle} = \sqrt{\left\langle \vec{v} | \vec{v} \right\rangle} \\ \left\|\vec{v}\right\|^2 &= \vec{v} \cdot \vec{v} = \left(\vec{v} \cdot \vec{v}\right)^{\frac{1}{2} \cdot 2} \\ &= \left(\vec{v}, \vec{v}\right) = \left\langle \vec{v}, \vec{v} \right\rangle = \left\langle \vec{v} | \vec{v} \right\rangle \end{aligned}$$

metric

$$d : X \times X \to \mathbb{R}$$

$$d(\cdot, \cdot) : X^2 \to \mathbb{R}$$

$d = d(\square, \square)$ is a metric distance function on the vector space $\mathcal{V}$

$\Updownarrow$

$$\begin{cases} d : X \times X \to \mathbb{R} & X \text{ is a nonempty set } \Leftrightarrow X \neq \emptyset \\ d(x, y) \geq 0, \quad d(x, y) = 0 \Leftrightarrow x = y \qquad \forall x, y \in X & (pd) \text{ positive definiteness} \\ d(x, y) = d(y, x) \qquad \forall x, y \in X & (s) \text{ symmetry} \\ d(x, y) \leq d(x, z) + d(z, y) \qquad \forall x, y, z \in X & (ti) \text{ subadditivity/triangle inequality} \end{cases}$$

$$\begin{aligned} d\left(\vec{u}, \vec{v}\right) &= \sqrt{\left(\vec{u} - \vec{v}\right) \cdot \left(\vec{u} - \vec{v}\right)} = +\sqrt{\left(\vec{u} - \vec{v}\right) \cdot \left(\vec{u} - \vec{v}\right)} = \left[\left(\vec{u} - \vec{v}\right) \cdot \left(\vec{u} - \vec{v}\right)\right]^{\frac{1}{2}} \\ &= \sqrt{\left(\vec{u} - \vec{v}, \vec{u} - \vec{v}\right)} = \sqrt{\left\langle \vec{u} - \vec{v}, \vec{u} - \vec{v} \right\rangle} = \sqrt{\left\langle \vec{u} - \vec{v} | \vec{u} - \vec{v} \right\rangle} \end{aligned}$$

$$\left\|\vec{v}\right\| = \sqrt{\vec{v} \cdot \vec{v}}$$

$$\left\|\vec{v}\right\|^2 = \vec{v} \cdot \vec{v}$$

invariant metric such as norm or inner product $\left\|\vec{v}\right\|^2 = \vec{v}\cdot\vec{v}$ but metric tensor component scalar

$$
\begin{aligned}
\left\|\vec{v}\right\|^2 =& \vec{v}\cdot\vec{v} \\
=& \left(v^1\vec{e}_1 + v^2\vec{e}_2\right)\cdot\left(v^1\vec{e}_1 + v^2\vec{e}_2\right) = \left(v^1\vec{e}_1\cdot + v^2\vec{e}_2\cdot\right)\left(v^1\vec{e}_1 + v^2\vec{e}_2\right) \\
=& v^1\vec{e}_1\cdot\left(v^1\vec{e}_1 + v^2\vec{e}_2\right) + v^2\vec{e}_2\cdot\left(v^1\vec{e}_1 + v^2\vec{e}_2\right) \\
=& v^1\vec{e}_1\cdot v^1\vec{e}_1 + v^1\vec{e}_1\cdot v^2\vec{e}_2 + v^2\vec{e}_2\cdot v^1\vec{e}_1 + v^2\vec{e}_2\cdot v^2\vec{e}_2 \\
=& v^1v^1\vec{e}_1\cdot\vec{e}_1 + v^1v^2\vec{e}_1\cdot\vec{e}_2 + v^2v^1\vec{e}_2\cdot\vec{e}_1 + v^2v^2\vec{e}_2\cdot\vec{e}_2 \\
=& v^1v^1\left(\vec{e}_1\cdot\vec{e}_1\right) + v^1v^2\left(\vec{e}_1\cdot\vec{e}_2\right) + v^2v^1\left(\vec{e}_2\cdot\vec{e}_1\right) + v^2v^2\left(\vec{e}_2\cdot\vec{e}_2\right) \\
=& v^{12}\left(\vec{e}_1\cdot\vec{e}_1\right) + 2\,v^1v^2\left(\vec{e}_1\cdot\vec{e}_2\right) + v^{22}\left(\vec{e}_2\cdot\vec{e}_2\right) && \vec{e}_1\cdot\vec{e}_2 = \vec{e}_2\cdot\vec{e}_1,\quad v^1v^2 = v^2v^1 \\
=& \left(\tilde{v}^1\tilde{\vec{e}}_1 + \tilde{v}^2\tilde{\vec{e}}_2\right)\cdot\left(\tilde{v}^1\tilde{\vec{e}}_1 + \tilde{v}^2\tilde{\vec{e}}_2\right) = \left(\tilde{v}^1\tilde{\vec{e}}_1\cdot + \tilde{v}^2\tilde{\vec{e}}_2\cdot\right)\left(\tilde{v}^1\tilde{\vec{e}}_1 + \tilde{v}^2\tilde{\vec{e}}_2\right) \\
=& \tilde{v}^{12}\left(\tilde{\vec{e}}_1\cdot\tilde{\vec{e}}_1\right) + 2\,\tilde{v}^1\tilde{v}^2\left(\tilde{\vec{e}}_1\cdot\tilde{\vec{e}}_2\right) + \tilde{v}^{22}\left(\tilde{\vec{e}}_2\cdot\tilde{\vec{e}}_2\right) \\
=& \left(v^1\vec{e}_1\cdot + v^2\vec{e}_2\cdot\right)\left(v^1\vec{e}_1 + v^2\vec{e}_2\right) = \begin{bmatrix} v^1 & v^2 \end{bmatrix}\begin{bmatrix}\vec{e}_1\cdot \\ \vec{e}_2\cdot\end{bmatrix}\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\begin{bmatrix}v^1 \\ v^2\end{bmatrix} \\
=& \begin{bmatrix} v^1 & v^2 \end{bmatrix}\begin{bmatrix}\vec{e}_1\cdot \\ \vec{e}_2\cdot\end{bmatrix}\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\begin{bmatrix}v^1 \\ v^2\end{bmatrix} = \begin{bmatrix} v^1 & v^2 \end{bmatrix}\begin{bmatrix}\vec{e}_1\cdot\vec{e}_1 & \vec{e}_1\cdot\vec{e}_2 \\ \vec{e}_2\cdot\vec{e}_1 & \vec{e}_2\cdot\vec{e}_2\end{bmatrix}\begin{bmatrix}v^1 \\ v^2\end{bmatrix} && = \begin{bmatrix} v^1 & v^2 \end{bmatrix} g \begin{bmatrix}v^1 \\ v^2\end{bmatrix} \\
=& \left(\tilde{v}^1\tilde{\vec{e}}_1\cdot + \tilde{v}^2\tilde{\vec{e}}_2\cdot\right)\left(\tilde{v}^1\tilde{\vec{e}}_1 + \tilde{v}^2\tilde{\vec{e}}_2\right) = \begin{bmatrix} \tilde{v}^1 & \tilde{v}^2 \end{bmatrix}\begin{bmatrix}\tilde{\vec{e}}_1\cdot \\ \tilde{\vec{e}}_2\cdot\end{bmatrix}\begin{bmatrix}\tilde{\vec{e}}_1 & \tilde{\vec{e}}_2\end{bmatrix}\begin{bmatrix}\tilde{v}^1 \\ \tilde{v}^2\end{bmatrix} \\
=& \begin{bmatrix} \tilde{v}^1 & \tilde{v}^2 \end{bmatrix}\begin{bmatrix}\tilde{\vec{e}}_1\cdot \\ \tilde{\vec{e}}_2\cdot\end{bmatrix}\begin{bmatrix}\tilde{\vec{e}}_1 & \tilde{\vec{e}}_2\end{bmatrix}\begin{bmatrix}\tilde{v}^1 \\ \tilde{v}^2\end{bmatrix} = \begin{bmatrix} \tilde{v}^1 & \tilde{v}^2 \end{bmatrix}\begin{bmatrix}\tilde{\vec{e}}_1\cdot\tilde{\vec{e}}_1 & \tilde{\vec{e}}_1\cdot\tilde{\vec{e}}_2 \\ \tilde{\vec{e}}_2\cdot\tilde{\vec{e}}_1 & \tilde{\vec{e}}_2\cdot\tilde{\vec{e}}_2\end{bmatrix}\begin{bmatrix}\tilde{v}^1 \\ \tilde{v}^2\end{bmatrix} && = \begin{bmatrix} \tilde{v}^1 & \tilde{v}^2 \end{bmatrix} \tilde{g} \begin{bmatrix}\tilde{v}^1 \\ \tilde{v}^2\end{bmatrix}
\end{aligned}
$$

$$
g = \begin{bmatrix}\vec{e}_1\cdot\vec{e}_1 & \vec{e}_1\cdot\vec{e}_2 \\ \vec{e}_2\cdot\vec{e}_1 & \vec{e}_2\cdot\vec{e}_2\end{bmatrix} \Leftrightarrow g_{ij} = \vec{e}_i\cdot\vec{e}_j
$$

$$
\tilde{g} = \begin{bmatrix}\tilde{\vec{e}}_1\cdot\tilde{\vec{e}}_1 & \tilde{\vec{e}}_1\cdot\tilde{\vec{e}}_2 \\ \tilde{\vec{e}}_2\cdot\tilde{\vec{e}}_1 & \tilde{\vec{e}}_2\cdot\tilde{\vec{e}}_2\end{bmatrix} \Leftrightarrow \tilde{g}_{ij} = \tilde{\vec{e}}_i\cdot\tilde{\vec{e}}_j
$$

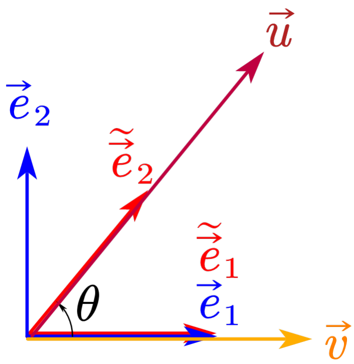

Figure 1.7: metric and angle

$$
\begin{cases}
\tilde{\vec{e}}_1 = \vec{e}_1 \\
\tilde{\vec{e}}_2 = \vec{e}_1\cos\theta + \vec{e}_2\sin\theta
\end{cases}
$$

$$\widetilde{g} = \begin{bmatrix} \widetilde{\vec{e}}_1\cdot\widetilde{\vec{e}}_1 & \widetilde{\vec{e}}_1\cdot\widetilde{\vec{e}}_2 \\ \widetilde{\vec{e}}_2\cdot\widetilde{\vec{e}}_1 & \widetilde{\vec{e}}_2\cdot\widetilde{\vec{e}}_2 \end{bmatrix} = \begin{bmatrix} \vec{e}_1\cdot\vec{e}_1 & \vec{e}_1\cdot\left(\vec{e}_1\cos\theta+\vec{e}_2\sin\theta\right) \\ \left(\vec{e}_1\cos\theta+\vec{e}_2\sin\theta\right)\cdot\vec{e}_1 & \left(\vec{e}_1\cos\theta+\vec{e}_2\sin\theta\right)\cdot\left(\vec{e}_1\cos\theta+\vec{e}_2\sin\theta\right) \end{bmatrix}$$
$$= \begin{bmatrix} 1 & \cos\theta \\ \cos\theta & \cos^2\theta+\sin^2\theta \end{bmatrix} = \begin{bmatrix} 1 & \cos\theta \\ \cos\theta & 1 \end{bmatrix}$$

$$\vec{u}\cdot\vec{v} = \left(\widetilde{u}\widetilde{\vec{e}}_2\right)\cdot\left(\widetilde{v}\widetilde{\vec{e}}_1\right) = \widetilde{u}\widetilde{v}\widetilde{\vec{e}}_2\cdot\widetilde{\vec{e}}_1 = \widetilde{u}\widetilde{v}\left(\widetilde{\vec{e}}_2\cdot\widetilde{\vec{e}}_1\right) = \widetilde{u}\widetilde{v}\cos\theta = \left\|\vec{u}\right\|\left\|\vec{v}\right\|\cos\theta$$

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\left\|\vec{u}\right\|\left\|\vec{v}\right\|}$$
$$\theta = \arccos\frac{\vec{u}\cdot\vec{v}}{\left\|\vec{u}\right\|\left\|\vec{v}\right\|}$$

$$\begin{aligned}
\left\|\vec{u}\right\|\left\|\vec{v}\right\|\cos\theta =& \vec{u}\cdot\vec{v} \\
=& \left(u^1\vec{e}_1+u^2\vec{e}_2\right)\cdot\left(v^1\vec{e}_1+v^2\vec{e}_2\right) = \left(u^1\vec{e}_1\cdot+u^2\vec{e}_2\cdot\right)\left(v^1\vec{e}_1+v^2\vec{e}_2\right) \\
=& u^1\vec{e}_1\cdot\left(v^1\vec{e}_1+v^2\vec{e}_2\right)+u^2\vec{e}_2\cdot\left(v^1\vec{e}_1+v^2\vec{e}_2\right) \\
=& u^1\vec{e}_1\cdot v^1\vec{e}_1+u^1\vec{e}_1\cdot v^2\vec{e}_2+u^2\vec{e}_2\cdot v^1\vec{e}_1+u^2\vec{e}_2\cdot v^2\vec{e}_2 \\
=& u^1v^1\vec{e}_1\cdot\vec{e}_1+u^1v^2\vec{e}_1\cdot\vec{e}_2+u^2v^1\vec{e}_2\cdot\vec{e}_1+u^2v^2\vec{e}_2\cdot\vec{e}_2 \\
=& u^1v^1\left(\vec{e}_1\cdot\vec{e}_1\right)+u^1v^2\left(\vec{e}_1\cdot\vec{e}_2\right)+u^2v^1\left(\vec{e}_2\cdot\vec{e}_1\right)+u^2v^2\left(\vec{e}_2\cdot\vec{e}_2\right) \\
=& u^1v^1\left(\vec{e}_1\cdot\vec{e}_1\right)+\left(u^1v^2+u^2v^1\right)\left(\vec{e}_1\cdot\vec{e}_2\right)+u^2v^2\left(\vec{e}_2\cdot\vec{e}_2\right) \qquad \vec{e}_1\cdot\vec{e}_2=\vec{e}_2\cdot\vec{e}_1 \\
=& \left(\widetilde{u}^1\widetilde{\vec{e}}_1+\widetilde{u}^2\widetilde{\vec{e}}_2\right)\cdot\left(\widetilde{v}^1\widetilde{\vec{e}}_1+\widetilde{v}^2\widetilde{\vec{e}}_2\right) = \left(\widetilde{u}^1\widetilde{\vec{e}}_1\cdot+\widetilde{u}^2\widetilde{\vec{e}}_2\cdot\right)\left(\widetilde{v}^1\widetilde{\vec{e}}_1+\widetilde{v}^2\widetilde{\vec{e}}_2\right) \\
=& u^1v^1\left(\vec{e}_1\cdot\vec{e}_1\right)+\left(u^1v^2+u^2v^1\right)\left(\vec{e}_1\cdot\vec{e}_2\right)+u^2v^2\left(\vec{e}_2\cdot\vec{e}_2\right) \\
=& \left(\widetilde{u}^1\widetilde{\vec{e}}_1\cdot+\widetilde{u}^2\widetilde{\vec{e}}_2\cdot\right)\left(\widetilde{v}^1\widetilde{\vec{e}}_1+\widetilde{v}^2\widetilde{\vec{e}}_2\right) = \begin{bmatrix}\widetilde{u}^1 & \widetilde{u}^2\end{bmatrix}\begin{bmatrix}\widetilde{\vec{e}}_1\cdot \\ \widetilde{\vec{e}}_2\cdot\end{bmatrix}\begin{bmatrix}\widetilde{\vec{e}}_1 & \widetilde{\vec{e}}_2\end{bmatrix}\begin{bmatrix}\widetilde{v}^1 \\ \widetilde{v}^2\end{bmatrix} \\
=& \begin{bmatrix}u^1 & u^2\end{bmatrix}\begin{bmatrix}\vec{e}_1\cdot \\ \vec{e}_2\cdot\end{bmatrix}\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\begin{bmatrix}v^1 \\ v^2\end{bmatrix} = \begin{bmatrix}u^1 & u^2\end{bmatrix}\begin{bmatrix}\vec{e}_1\cdot\vec{e}_1 & \vec{e}_1\cdot\vec{e}_2 \\ \vec{e}_2\cdot\vec{e}_1 & \vec{e}_2\cdot\vec{e}_2\end{bmatrix}\begin{bmatrix}v^1 \\ v^2\end{bmatrix} \qquad = \begin{bmatrix}u^1 & u^2\end{bmatrix} g \begin{bmatrix}v^1 \\ v^2\end{bmatrix} \\
=& \left(\widetilde{u}^1\widetilde{\vec{e}}_1\cdot+\widetilde{u}^2\widetilde{\vec{e}}_2\cdot\right)\left(\widetilde{v}^1\widetilde{\vec{e}}_1+\widetilde{v}^2\widetilde{\vec{e}}_2\right) = \begin{bmatrix}\widetilde{u}^1 & \widetilde{u}^2\end{bmatrix}\begin{bmatrix}\widetilde{\vec{e}}_1\cdot \\ \widetilde{\vec{e}}_2\cdot\end{bmatrix}\begin{bmatrix}\widetilde{\vec{e}}_1 & \widetilde{\vec{e}}_2\end{bmatrix}\begin{bmatrix}\widetilde{v}^1 \\ \widetilde{v}^2\end{bmatrix} \\
=& \begin{bmatrix}\widetilde{u}^1 & \widetilde{u}^2\end{bmatrix}\begin{bmatrix}\widetilde{\vec{e}}_1\cdot \\ \widetilde{\vec{e}}_2\cdot\end{bmatrix}\begin{bmatrix}\widetilde{\vec{e}}_1 & \widetilde{\vec{e}}_2\end{bmatrix}\begin{bmatrix}\widetilde{v}^1 \\ \widetilde{v}^2\end{bmatrix} = \begin{bmatrix}\widetilde{u}^1 & \widetilde{u}^2\end{bmatrix}\begin{bmatrix}\widetilde{\vec{e}}_1\cdot\widetilde{\vec{e}}_1 & \widetilde{\vec{e}}_1\cdot\widetilde{\vec{e}}_2 \\ \widetilde{\vec{e}}_2\cdot\widetilde{\vec{e}}_1 & \widetilde{\vec{e}}_2\cdot\widetilde{\vec{e}}_2\end{bmatrix}\begin{bmatrix}\widetilde{v}^1 \\ \widetilde{v}^2\end{bmatrix} \qquad = \begin{bmatrix}\widetilde{u}^1 & \widetilde{u}^2\end{bmatrix}\widetilde{g}\begin{bmatrix}\widetilde{v}^1 \\ \widetilde{v}^2\end{bmatrix}
\end{aligned}$$

$(0,1)$-tensor
covariant
basis vector

$$\begin{cases}\widetilde{\vec{e}}_j = \vec{e}_i\widetilde{e}^i_j \\ \vec{e}_j = \widetilde{\vec{e}}_i e^i_j\end{cases}$$

$(1,0)$-tensor
contravariant
vector component scalar

$$\begin{cases}\widetilde{v}^i = e^i_j v^j \\ v^i = \widetilde{e}^i_j\widetilde{v}^j\end{cases}$$

$(1,1)$-tensor

covector component scalar

$$\begin{cases}\widetilde{\alpha}_j = \alpha_i\widetilde{e}^i_j \\ \alpha_j = \widetilde{\alpha}_i e^i_j\end{cases}$$

dual basis covector

$$\begin{cases}\widetilde{\overleftarrow{\epsilon}}^i = e^i_j\overleftarrow{\epsilon}^j \\ \overleftarrow{\epsilon}^i = \widetilde{e}^i_j\widetilde{\overleftarrow{\epsilon}}^j\end{cases}$$

linear map

$$\begin{cases}\widetilde{L}^i_j = e^i_h L^h_k\widetilde{e}^k_j \\ L^i_j = \widetilde{e}^i_h\widetilde{L}^h_k e^k_j\end{cases}$$

$$\begin{cases} \overset{\to}{\widetilde{e}}_i = \vec{e}_h \widetilde{e}^h_i \\ \overset{\to}{\widetilde{e}}_j = \vec{e}_k \widetilde{e}^k_j \end{cases} \Leftarrow \overset{\to}{\widetilde{e}}_j = \vec{e}_i \widetilde{e}^i_j$$

$$\begin{aligned} \widetilde{g}_{ij} = \overset{\to}{\widetilde{e}}_i \cdot \overset{\to}{\widetilde{e}}_j &= \left(\vec{e}_h \widetilde{e}^h_i\right) \cdot \left(\vec{e}_k \widetilde{e}^k_j\right) \\ &= \left(\vec{e}_h \cdot \vec{e}_k\right) \widetilde{e}^h_i \widetilde{e}^k_j \qquad = \left(\vec{e}_k \cdot \vec{e}_h\right) \widetilde{e}^k_j \widetilde{e}^h_i \qquad g_{hk} = \left(\vec{e}_h \cdot \vec{e}_k\right) = \left(\vec{e}_k \cdot \vec{e}_h\right) = g_{kh} \\ &= g_{hk} \widetilde{e}^h_i \widetilde{e}^k_j \qquad\qquad g_{hk} = \vec{e}_h \cdot \vec{e}_k \\ \widetilde{g}_{ij} &= g_{hk} \widetilde{e}^h_i \widetilde{e}^k_j \\ &\Downarrow \\ g_{ij} &= \widetilde{g}_{hk} e^h_i e^k_j \end{aligned}$$

$(0,1)$-tensor	$(1,0)$-tensor	
covariant	contravariant	
basis vector	vector component scalar	$(1,1)$-tensor
$\begin{cases} \overset{\to}{\widetilde{e}}_j = \vec{e}_i \widetilde{e}^i_j \\ \vec{e}_j = \overset{\to}{\widetilde{e}}_i e^i_j \end{cases}$	$\begin{cases} \widetilde{v}^i = e^i_j v^j \\ v^i = \widetilde{e}^i_j \widetilde{v}^j \end{cases}$	
covector component scalar	dual basis covector	linear map
$\begin{cases} \widetilde{\alpha}_j = \alpha_i \widetilde{e}^i_j \\ \alpha_j = \widetilde{\alpha}_i e^i_j \end{cases}$	$\begin{cases} \overset{\leftarrow}{\widetilde{\epsilon}}^i = e^i_j \overset{\leftarrow}{\epsilon}^j \\ \overset{\leftarrow}{\epsilon}^i = \widetilde{e}^i_j \overset{\leftarrow}{\widetilde{\epsilon}}^j \end{cases}$	$\begin{cases} \widetilde{L}^i_j = e^i_h L^h_k \widetilde{e}^k_j \\ L^i_j = \widetilde{e}^i_h \widetilde{L}^h_k e^k_j \end{cases}$
$(0,2)$-tensor		
metric tensor		
$\begin{cases} \widetilde{g}_{ij} = g_{hk} \widetilde{e}^h_i \widetilde{e}^k_j \\ g_{ij} = \widetilde{g}_{hk} e^h_i e^k_j \end{cases}$		

Dirac notation = Dirac bra-ket notation = Dirac bracket notation, usually used in complete inner product [vector] space = Hilbert space

$$\begin{aligned} \vec{v}_i \cdot \vec{v} &= \left(\vec{v}_i, \vec{v}\right) = \left\langle \vec{v}_i, \vec{v} \right\rangle = g\left(\vec{v}_i, \vec{v}\right) \\ &= \vec{v}_i 1 \cdot \vec{v} = \left(\vec{v}_i 1, \vec{v}\right) = \left\langle \vec{v}_i 1, \vec{v} \right\rangle = g\left(\vec{v}_i 1, \vec{v}\right) \\ &= \left(\vec{v}_i 1\right) \cdot \left(\vec{v}_j v^j\right) = \left(\vec{v}_i \cdot \vec{v}_j\right) 1 v^j = g_{ij} 1 v^j = 1 \left(\vec{v}_i \cdot \vec{v}_j\right) v^j = 1 g_{ij} v^j \\ &= \left(\vec{v}_i\right) \cdot \left(\vec{v}_j v^j\right) = \left(\vec{v}_i \cdot \vec{v}_j\right) v^j = g_{ij} v^j = \left(\vec{v}_i \cdot \vec{v}_j\right) v^j = g_{ij} v^j \end{aligned}$$

$$\begin{aligned} \vec{e}_i \cdot \vec{v} &= \left(\vec{e}_i, \vec{v}\right) = \left\langle \vec{e}_i, \vec{v} \right\rangle \\ &= \left(\vec{e}_i\right) \cdot \left(\vec{e}_j v^j\right) = \left(\vec{e}_i \cdot \vec{e}_j\right) v^j = \delta_{ij} v^j = v^i \qquad g_{ij} = \vec{e}_i \cdot \vec{e}_j = \delta_{ij} \\ v^i &= \vec{e}_i \cdot \vec{v} = \left(\vec{e}_i, \vec{v}\right) = \left\langle \vec{e}_i, \vec{v} \right\rangle \end{aligned}$$

$$v^i = \vec{e}_i \cdot \vec{v} = \left(\vec{e}_i, \vec{v}\right) = \left\langle \vec{e}_i, \vec{v} \right\rangle$$

$$\begin{aligned} \vec{v} &= \sum_j \vec{e}_j v^j = \vec{e}_j v^j \\ &= \sum_j \vec{e}_j \left(\vec{e}_j \cdot \vec{v}\right) = \vec{e}_j \left(\vec{e}_j \cdot \vec{v}\right) \qquad = \sum_j \vec{e}_j \left(\vec{e}_j, \vec{v}\right) = \vec{e}_j \left(\vec{e}_j, \vec{v}\right) = \sum_j \vec{e}_j \left\langle \vec{e}_j, \vec{v} \right\rangle = \vec{e}_j \left\langle \vec{e}_j, \vec{v} \right\rangle \\ \vec{v} &= \sum_j \vec{e}_j \left\langle \vec{e}_j, \vec{v} \right\rangle = \vec{e}_j \left\langle \vec{e}_j, \vec{v} \right\rangle \end{aligned}$$

$$
\begin{aligned}
\vec{v}\cdot &= \left(\vec{v},\cdot\right) = \left\langle \vec{v},\cdot\right\rangle \\
= \vec{v}\cdot\square &= \left(\vec{v},\square\right) = \left\langle \vec{v},\square\right\rangle \\
&= \left\langle \sum_i \vec{e}_i \left\langle \vec{e}_i,\vec{v}\right\rangle,\square\right\rangle = \left\langle \vec{e}_i \left\langle \vec{e}_i,\vec{v}\right\rangle,\square\right\rangle \\
&= \left\langle \sum_i \left\langle \vec{e}_i,\vec{v}\right\rangle \vec{e}_i,\square\right\rangle = \left\langle \left\langle \vec{e}_i,\vec{v}\right\rangle \vec{e}_i,\square\right\rangle \\
&\overset{\text{antilinearity}}{=} \overline{\sum_i \left\langle \vec{e}_i,\vec{v}\right\rangle} \left\langle \vec{e}_i,\square\right\rangle = \overline{\left\langle \vec{e}_i,\vec{v}\right\rangle} \left\langle \vec{e}_i,\square\right\rangle \\
&= \sum_i \overline{\left\langle \vec{e}_i,\vec{v}\right\rangle} \left\langle \vec{e}_i,\square\right\rangle = \overline{\left\langle \vec{e}_i,\vec{v}\right\rangle} \left\langle \vec{e}_i,\square\right\rangle \\
&= \sum_i \left\langle \vec{v},\vec{e}_i\right\rangle \left\langle \vec{e}_i,\square\right\rangle = \left\langle \vec{v},\vec{e}_i\right\rangle \left\langle \vec{e}_i,\square\right\rangle
\end{aligned}
$$

$$\vec{v} = \sum_j \vec{e}_j \left\langle \vec{e}_j,\vec{v}\right\rangle = \vec{e}_j \left\langle \vec{e}_j,\vec{v}\right\rangle$$

$$\vec{v} = \sum_i \vec{e}_i \left\langle \vec{e}_i,\vec{v}\right\rangle = \vec{e}_i \left\langle \vec{e}_i,\vec{v}\right\rangle$$

$$\overline{a+bi} = (a+bi)^* = a-bi$$

$$\left\langle \vec{u}|\vec{v}\right\rangle = \left\langle \vec{v}|\vec{u}\right\rangle^* = \overline{\left\langle \vec{v}|\vec{u}\right\rangle}$$

$$\vec{v} = \sum_j \vec{e}_j \left\langle \vec{e}_j,\vec{v}\right\rangle = \vec{e}_j \left\langle \vec{e}_j,\vec{v}\right\rangle$$

$$\vec{v}\cdot\square = \left\langle \vec{v},\square\right\rangle = \sum_i \left\langle \vec{v},\vec{e}_i\right\rangle \left\langle \vec{e}_i,\square\right\rangle = \left\langle \vec{v},\vec{e}_i\right\rangle \left\langle \vec{e}_i,\square\right\rangle$$

↓Dirac bra-ket notation

$$\left|\vec{v}\right\rangle = \sum_j \left|\vec{e}_j\right\rangle \left\langle \vec{e}_j|\vec{v}\right\rangle = \left|\vec{e}_j\right\rangle \left\langle \vec{e}_j|\vec{v}\right\rangle$$

$$
\begin{aligned}
\vec{v}\cdot\square = \left\langle \vec{v}|\square\right\rangle = \left\langle \vec{v}\right| &= \sum_i \left\langle \vec{v}|\vec{e}_i\right\rangle \left\langle \vec{e}_i|\square\right\rangle = \left\langle \vec{v}|\vec{e}_i\right\rangle \left\langle \vec{e}_i|\square\right\rangle \\
&= \sum_i \left\langle \vec{v}|\vec{e}_i\right\rangle \left\langle \vec{e}_i\right| = \left\langle \vec{v}|\vec{e}_i\right\rangle \left\langle \vec{e}_i\right|
\end{aligned}
$$

$$\left|\vec{v}\right\rangle = \sum_j \left|\vec{e}_j\right\rangle \left\langle \vec{e}_j|\vec{v}\right\rangle = \left|\vec{e}_j\right\rangle \left\langle \vec{e}_j|\vec{v}\right\rangle = \sum_j \left\langle \vec{e}_j|\vec{v}\right\rangle \left|\vec{e}_j\right\rangle = \left\langle \vec{e}_j|\vec{v}\right\rangle \left|\vec{e}_j\right\rangle$$

$$\left\langle \vec{v}\right| = \sum_i \left\langle \vec{v}|\vec{e}_i\right\rangle \left\langle \vec{e}_i\right| = \left\langle \vec{v}|\vec{e}_i\right\rangle \left\langle \vec{e}_i\right| = \sum_i \left\langle \vec{e}_i\right| \left\langle \vec{v}|\vec{e}_i\right\rangle = \left\langle \vec{e}_i\right| \left\langle \vec{v}|\vec{e}_i\right\rangle$$

$$\left\langle \vec{v}\right| \left|\vec{u}\right\rangle = \begin{cases} \left\langle \vec{v}|\vec{u}\right\rangle = \left\langle \vec{e}_i v^i|\vec{e}_j u^j\right\rangle = v^i \left\langle \vec{e}_i|\vec{e}_j\right\rangle u^j = v^i \vec{e}_i\cdot\vec{e}_j u^j = v^i \left(\vec{e}_i\cdot\vec{e}_j\right) u^j = v^i g_{ij} u^j \\ \left\langle \overleftarrow{v}|\vec{u}\right\rangle = \left\langle v_i \overleftarrow{\epsilon}^i|\vec{e}_j u^j\right\rangle = v_i \left\langle \overleftarrow{\epsilon}^i|\vec{e}_j\right\rangle u^j = v_i \overleftarrow{\epsilon}^i \vec{e}_j u^j = g_{ij} v^j \overleftarrow{\epsilon}^i \vec{e}_j u^j = g_{ij} v^j \delta^i{}_j u^j = g_{ij} v^i u^j = v^i g_{ij} u^j \end{cases}$$

$$
\begin{aligned}
\begin{bmatrix}u^1 & u^2\end{bmatrix} g \begin{bmatrix}v^1\\ v^2\end{bmatrix} &= \begin{bmatrix}u^1 & u^2\end{bmatrix}\begin{bmatrix}\vec{e}_1\cdot\vec{e}_1 & \vec{e}_1\cdot\vec{e}_2\\ \vec{e}_2\cdot\vec{e}_1 & \vec{e}_2\cdot\vec{e}_2\end{bmatrix}\begin{bmatrix}v^1\\ v^2\end{bmatrix} = \begin{bmatrix}u^1 & u^2\end{bmatrix}\begin{bmatrix}g_{11} & g_{12}\\ g_{21} & g_{22}\end{bmatrix}\begin{bmatrix}v^1\\ v^2\end{bmatrix}\\
&= \begin{bmatrix}u^1 & u^2\end{bmatrix}\begin{bmatrix}\vec{e}_1\cdot\\ \vec{e}_2\cdot\end{bmatrix}\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\begin{bmatrix}v^1\\ v^2\end{bmatrix}\\
&= \left(u^1\vec{e}_1\cdot + u^2\vec{e}_2\cdot\right)\left(\vec{e}_1 v^1 + \vec{e}_2 v^2\right)\\
&= u^1\vec{e}_1\cdot\left(\vec{e}_1 v^1 + \vec{e}_2 v^2\right) + u^2\vec{e}_2\cdot\left(\vec{e}_1 v^1 + \vec{e}_2 v^2\right)\\
&= \left(u^1\vec{e}_1\cdot\vec{e}_1 v^1 + u^1\vec{e}_1\cdot\vec{e}_2 v^2\right) + \left(u^2\vec{e}_2\cdot\vec{e}_1 v^1 + u^2\vec{e}_2\cdot\vec{e}_2 v^2\right)\\
&= u^1 v^1\vec{e}_1\cdot\vec{e}_1 + u^1 v^2\vec{e}_1\cdot\vec{e}_2 + u^2 v^1\vec{e}_2\cdot\vec{e}_1 + u^2 v^2\vec{e}_2\cdot\vec{e}_2\\
&= u^1\vec{e}_1\cdot\vec{e}_1 v^1 + u^1\vec{e}_1\cdot\vec{e}_2 v^2 + u^2\vec{e}_2\cdot\vec{e}_1 v^1 + u^2\vec{e}_2\cdot\vec{e}_2 v^2\\
&= \vec{e}_1\cdot\vec{e}_1 u^1 v^1 + \vec{e}_1\cdot\vec{e}_2 u^1 v^2 + \vec{e}_2\cdot\vec{e}_1 u^2 v^1 + \vec{e}_2\cdot\vec{e}_2 u^2 v^2\\
&= u^1 g_{11} v^1 + u^1 g_{12} v^2 + u^2 g_{21} v^1 + u^2 g_{22} v^2\\
&= g_{11} u^1 v^1 + g_{12} u^1 v^2 + g_{21} u^2 v^1 + g_{22} u^2 v^2\\
= g\begin{bmatrix}u^1\\ u^2\end{bmatrix}\begin{bmatrix}v^1\\ v^2\end{bmatrix} &= \begin{bmatrix}\begin{bmatrix}g_{11} & g_{12}\end{bmatrix} & \begin{bmatrix}g_{21} & g_{22}\end{bmatrix}\end{bmatrix}\begin{bmatrix}u^1\\ u^2\end{bmatrix}\begin{bmatrix}v^1\\ v^2\end{bmatrix} = \left(\begin{bmatrix}g_{11} & g_{12}\end{bmatrix}u^1 + \begin{bmatrix}g_{21} & g_{22}\end{bmatrix}u^2\right)\begin{bmatrix}v^1\\ v^2\end{bmatrix}\\
&= \left(\begin{bmatrix}g_{11}u^1 & g_{12}u^1\end{bmatrix} + \begin{bmatrix}g_{21}u^2 & g_{22}u^2\end{bmatrix}\right)\begin{bmatrix}v^1\\ v^2\end{bmatrix} = \begin{bmatrix}g_{11}u^1 + g_{21}u^2 & g_{12}u^1 + g_{22}u^2\end{bmatrix}\begin{bmatrix}v^1\\ v^2\end{bmatrix}\\
&= \left(g_{11}u^1 + g_{21}u^2\right)v^1 + \left(g_{12}u^1 + g_{22}u^2\right)v^2 = \left(g_{11}u^1 v^1 + g_{21}u^2 v^1\right) + \left(g_{12}u^1 v^2 + g_{22}u^2 v^2\right)\\
&= g_{11}u^1 v^1 + g_{21}u^2 v^1 + g_{12}u^1 v^2 + g_{22}u^2 v^2\\
&= \begin{bmatrix}\begin{bmatrix}g_{11} & g_{12}\end{bmatrix} & \begin{bmatrix}g_{21} & g_{22}\end{bmatrix}\end{bmatrix}\begin{bmatrix}u^1\\ u^2\end{bmatrix}\begin{bmatrix}v^1\\ v^2\end{bmatrix} = \begin{bmatrix}\begin{bmatrix}\vec{e}_1\cdot\vec{e}_1 & \vec{e}_1\cdot\vec{e}_2\end{bmatrix} & \begin{bmatrix}\vec{e}_2\cdot\vec{e}_1 & \vec{e}_2\cdot\vec{e}_2\end{bmatrix}\end{bmatrix}\begin{bmatrix}u^1\\ u^2\end{bmatrix}\begin{bmatrix}v^1\\ v^2\end{bmatrix}\\
&= \begin{bmatrix}\begin{bmatrix}\vec{e}_1\cdot\vec{e}_1 & \vec{e}_1\cdot\vec{e}_2\end{bmatrix} & \begin{bmatrix}\vec{e}_2\cdot\vec{e}_1 & \vec{e}_2\cdot\vec{e}_2\end{bmatrix}\end{bmatrix}\begin{bmatrix}u^1\\ u^2\end{bmatrix}\begin{bmatrix}v^1\\ v^2\end{bmatrix} = \begin{bmatrix}\vec{e}_1\cdot\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix} & \vec{e}_2\cdot\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\end{bmatrix}\begin{bmatrix}u^1\\ u^2\end{bmatrix}\begin{bmatrix}v^1\\ v^2\end{bmatrix}\\
&= \begin{bmatrix}\vec{e}_1\cdot & \vec{e}_2\cdot\end{bmatrix}\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\begin{bmatrix}u^1\\ u^2\end{bmatrix}\begin{bmatrix}v^1\\ v^2\end{bmatrix}
\end{aligned}
$$

$$
\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix} = \begin{bmatrix}\tilde{\vec{e}}_1 & \tilde{\vec{e}}_2\end{bmatrix}\begin{bmatrix}e^1_1 & e^1_2\\ e^2_1 & e^2_2\end{bmatrix}
$$

$$
\begin{bmatrix}\tilde{\vec{e}}_1 & \tilde{\vec{e}}_2\end{bmatrix} = \begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\begin{bmatrix}\tilde{e}^1_1 & \tilde{e}^1_2\\ \tilde{e}^2_1 & \tilde{e}^2_2\end{bmatrix}
$$

$$
\begin{bmatrix}\vec{e}_1\cdot\\ \vec{e}_2\cdot\end{bmatrix} = \begin{bmatrix}\vec{e}_1\\ \vec{e}_2\end{bmatrix}\cdot = \begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}^{\mathsf{T}}\cdot = \left(\begin{bmatrix}\tilde{\vec{e}}_1 & \tilde{\vec{e}}_2\end{bmatrix}\begin{bmatrix}e^1_1 & e^1_2\\ e^2_1 & e^2_2\end{bmatrix}\right)^{\mathsf{T}}\cdot = \begin{bmatrix}e^1_1 & e^1_2\\ e^2_1 & e^2_2\end{bmatrix}^{\mathsf{T}}\begin{bmatrix}\tilde{\vec{e}}_1 & \tilde{\vec{e}}_2\end{bmatrix}^{\mathsf{T}}\cdot = \begin{bmatrix}e^1_1 & e^1_2\\ e^2_1 & e^2_2\end{bmatrix}^{\mathsf{T}}\begin{bmatrix}\tilde{\vec{e}}_1\\ \tilde{\vec{e}}_2\end{bmatrix}\cdot = \begin{bmatrix}e^1_1 & e^1_2\\ e^2_1 & e^2_2\end{bmatrix}^{\mathsf{T}}\begin{bmatrix}\tilde{\vec{e}}_1\cdot\\ \tilde{\vec{e}}_2\cdot\end{bmatrix}
$$

$$
\begin{bmatrix}\tilde{\vec{e}}_1\cdot\\ \tilde{\vec{e}}_2\cdot\end{bmatrix} = \begin{bmatrix}\tilde{\vec{e}}_1\\ \tilde{\vec{e}}_2\end{bmatrix}\cdot = \begin{bmatrix}\tilde{\vec{e}}_1 & \tilde{\vec{e}}_2\end{bmatrix}^{\mathsf{T}}\cdot = \left(\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\begin{bmatrix}\tilde{e}^1_1 & \tilde{e}^1_2\\ \tilde{e}^2_1 & \tilde{e}^2_2\end{bmatrix}\right)^{\mathsf{T}}\cdot = \begin{bmatrix}\tilde{e}^1_1 & \tilde{e}^1_2\\ \tilde{e}^2_1 & \tilde{e}^2_2\end{bmatrix}^{\mathsf{T}}\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}^{\mathsf{T}}\cdot = \begin{bmatrix}\tilde{e}^1_1 & \tilde{e}^1_2\\ \tilde{e}^2_1 & \tilde{e}^2_2\end{bmatrix}^{\mathsf{T}}\begin{bmatrix}\vec{e}_1\\ \vec{e}_2\end{bmatrix}\cdot = \begin{bmatrix}\tilde{e}^1_1 & \tilde{e}^1_2\\ \tilde{e}^2_1 & \tilde{e}^2_2\end{bmatrix}^{\mathsf{T}}\begin{bmatrix}\vec{e}_1\cdot\\ \vec{e}_2\cdot\end{bmatrix}
$$

$$
\begin{bmatrix}\vec{e}_1\cdot\\ \vec{e}_2\cdot\end{bmatrix} = \begin{bmatrix}e^1_1 & e^1_2\\ e^2_1 & e^2_2\end{bmatrix}^{\mathsf{T}}\begin{bmatrix}\tilde{\vec{e}}_1\cdot\\ \tilde{\vec{e}}_2\cdot\end{bmatrix}
$$

$$
\begin{bmatrix}\tilde{\vec{e}}_1\cdot\\ \tilde{\vec{e}}_2\cdot\end{bmatrix} = \begin{bmatrix}\tilde{e}^1_1 & \tilde{e}^1_2\\ \tilde{e}^2_1 & \tilde{e}^2_2\end{bmatrix}^{\mathsf{T}}\begin{bmatrix}\vec{e}_1\cdot\\ \vec{e}_2\cdot\end{bmatrix}
$$

$$
\begin{bmatrix}\tilde{\vec{e}}_1 & \tilde{\vec{e}}_2\end{bmatrix} = \begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\begin{bmatrix}\tilde{e}^1_1 & \tilde{e}^1_2\\ \tilde{e}^2_1 & \tilde{e}^2_2\end{bmatrix} \qquad \begin{bmatrix}\tilde{\vec{e}}_1\cdot\\ \tilde{\vec{e}}_2\cdot\end{bmatrix} = \begin{bmatrix}\tilde{e}^1_1 & \tilde{e}^1_2\\ \tilde{e}^2_1 & \tilde{e}^2_2\end{bmatrix}^{\mathsf{T}}\begin{bmatrix}\vec{e}_1\cdot\\ \vec{e}_2\cdot\end{bmatrix}
$$

$$
\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix} = \begin{bmatrix}\tilde{\vec{e}}_1 & \tilde{\vec{e}}_2\end{bmatrix}\begin{bmatrix}e^1_1 & e^1_2\\ e^2_1 & e^2_2\end{bmatrix} \qquad \begin{bmatrix}\vec{e}_1\cdot\\ \vec{e}_2\cdot\end{bmatrix} = \begin{bmatrix}e^1_1 & e^1_2\\ e^2_1 & e^2_2\end{bmatrix}^{\mathsf{T}}\begin{bmatrix}\tilde{\vec{e}}_1\cdot\\ \tilde{\vec{e}}_2\cdot\end{bmatrix}
$$

$$
\begin{aligned}
g \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} &= \begin{bmatrix} \begin{bmatrix} g_{11} & g_{12} \end{bmatrix} & \begin{bmatrix} g_{21} & g_{22} \end{bmatrix} \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \left(\begin{bmatrix} g_{11} & g_{12} \end{bmatrix} u^1 + \begin{bmatrix} g_{21} & g_{22} \end{bmatrix} u^2 \right) \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \\
&= \left(\begin{bmatrix} g_{11} u^1 & g_{12} u^1 \end{bmatrix} + \begin{bmatrix} g_{21} u^2 & g_{22} u^2 \end{bmatrix} \right) \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \begin{bmatrix} g_{11} u^1 + g_{21} u^2 & g_{12} u^1 + g_{22} u^2 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \\
&= \left(g_{11} u^1 + g_{21} u^2 \right) v^1 + \left(g_{12} u^1 + g_{22} u^2 \right) v^2 = \left(g_{11} u^1 v^1 + g_{21} u^2 v^1 \right) + \left(g_{12} u^1 v^2 + g_{22} u^2 v^2 \right) \\
&= g_{11} u^1 v^1 + g_{21} u^2 v^1 + g_{12} u^1 v^2 + g_{22} u^2 v^2 \\
&= \begin{bmatrix} \begin{bmatrix} g_{11} & g_{12} \end{bmatrix} & \begin{bmatrix} g_{21} & g_{22} \end{bmatrix} \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \vec{e}_1 \cdot \vec{e}_1 & \vec{e}_1 \cdot \vec{e}_2 \end{bmatrix} & \begin{bmatrix} \vec{e}_2 \cdot \vec{e}_1 & \vec{e}_2 \cdot \vec{e}_2 \end{bmatrix} \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \\
&= \begin{bmatrix} \begin{bmatrix} \vec{e}_1 \cdot \vec{e}_1 & \vec{e}_1 \cdot \vec{e}_2 \end{bmatrix} & \begin{bmatrix} \vec{e}_2 \cdot \vec{e}_1 & \vec{e}_2 \cdot \vec{e}_2 \end{bmatrix} \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \begin{bmatrix} \vec{e}_1 \cdot \begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix} & \vec{e}_2 \cdot \begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix} \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \\
&= \begin{bmatrix} \vec{e}_1 \cdot \begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix} & \vec{e}_2 \cdot \begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix} \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \begin{bmatrix} \vec{e}_1 \cdot & \vec{e}_2 \cdot \end{bmatrix} \begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \\
&= g_{11} u^1 v^1 + g_{21} u^2 v^1 + g_{12} u^1 v^2 + g_{22} u^2 v^2 = \begin{bmatrix} \begin{bmatrix} g_{11} & g_{12} \end{bmatrix} & \begin{bmatrix} g_{21} & g_{22} \end{bmatrix} \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \\
&= g_{11} u^1 v^1 + g_{12} u^2 v^1 + g_{21} u^1 v^2 + g_{22} u^2 v^2 \qquad g_{21} = g_{12} \\
&= g_{11} u^1 v^1 + g_{21} u^1 v^2 + g_{12} u^2 v^1 + g_{22} u^2 v^2 = \begin{bmatrix} \begin{bmatrix} g_{11} & g_{21} \end{bmatrix} & \begin{bmatrix} g_{12} & g_{22} \end{bmatrix} \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \\
&= \begin{bmatrix} \begin{bmatrix} g_{11} & g_{21} \end{bmatrix} & \begin{bmatrix} g_{12} & g_{22} \end{bmatrix} \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \vec{e}_1 \cdot \vec{e}_1 & \vec{e}_2 \cdot \vec{e}_1 \end{bmatrix} & \begin{bmatrix} \vec{e}_1 \cdot \vec{e}_2 & \vec{e}_2 \cdot \vec{e}_2 \end{bmatrix} \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \\
&= \begin{bmatrix} \begin{bmatrix} \vec{e}_1 \cdot \vec{e}_1 & \vec{e}_2 \cdot \vec{e}_1 \end{bmatrix} & \begin{bmatrix} \vec{e}_1 \cdot \vec{e}_2 & \vec{e}_2 \cdot \vec{e}_2 \end{bmatrix} \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \vec{e}_1 \cdot & \vec{e}_2 \cdot \end{bmatrix} \vec{e}_1 & \begin{bmatrix} \vec{e}_1 \cdot & \vec{e}_2 \cdot \end{bmatrix} \vec{e}_2 \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \\
&= \begin{bmatrix} \begin{bmatrix} \vec{e}_1 \cdot & \vec{e}_2 \cdot \end{bmatrix} \vec{e}_1 & \begin{bmatrix} \vec{e}_1 \cdot & \vec{e}_2 \cdot \end{bmatrix} \vec{e}_2 \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \begin{bmatrix} \vec{e}_1 \cdot & \vec{e}_2 \cdot \end{bmatrix} \begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&\begin{bmatrix} \vec{e}_1 \cdot & \vec{e}_2 \cdot \end{bmatrix} \begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \\
&= \begin{cases} \begin{bmatrix} \begin{bmatrix} \vec{e}_1 \cdot & \vec{e}_2 \cdot \end{bmatrix} \vec{e}_1 & \begin{bmatrix} \vec{e}_1 \cdot & \vec{e}_2 \cdot \end{bmatrix} \vec{e}_2 \end{bmatrix} & \text{right distribution} \\ \begin{bmatrix} \vec{e}_1 \cdot \begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix} & \vec{e}_2 \cdot \begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix} \end{bmatrix} & \text{left distribution} \end{cases} \begin{bmatrix} u^1 \\ u^2 \\ \hline u^1 \\ u^2 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \\ \hline v^1 \\ v^2 \end{bmatrix} \\
&= \begin{cases} \begin{bmatrix} \begin{bmatrix} \vec{e}_1 \cdot \vec{e}_1 & \vec{e}_2 \cdot \vec{e}_1 \end{bmatrix} & \begin{bmatrix} \vec{e}_1 \cdot \vec{e}_2 & \vec{e}_2 \cdot \vec{e}_2 \end{bmatrix} \end{bmatrix} \\ \begin{bmatrix} \begin{bmatrix} \vec{e}_1 \cdot \vec{e}_1 & \vec{e}_1 \cdot \vec{e}_2 \end{bmatrix} & \begin{bmatrix} \vec{e}_2 \cdot \vec{e}_1 & \vec{e}_2 \cdot \vec{e}_2 \end{bmatrix} \end{bmatrix} \end{cases} \begin{bmatrix} u^1 \\ u^2 \\ \hline u^1 \\ u^2 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \\ \hline v^1 \\ v^2 \end{bmatrix} \\
&= \begin{cases} \begin{bmatrix} \begin{bmatrix} g_{11} & g_{21} \end{bmatrix} & \begin{bmatrix} g_{12} & g_{22} \end{bmatrix} \end{bmatrix} & \text{right distribution} \\ \begin{bmatrix} \begin{bmatrix} g_{11} & g_{12} \end{bmatrix} & \begin{bmatrix} g_{21} & g_{22} \end{bmatrix} \end{bmatrix} & \text{left distribution} \end{cases} \begin{bmatrix} u^1 \\ u^2 \\ \hline u^1 \\ u^2 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \\ \hline v^1 \\ v^2 \end{bmatrix} \qquad\qquad g_{ij} = \vec{e}_i \cdot \vec{e}_j \\
&= \begin{cases} \left(\begin{bmatrix} g_{11} & g_{21} \end{bmatrix} u^1 + \begin{bmatrix} g_{12} & g_{22} \end{bmatrix} u^2 \right) & = \left(\begin{bmatrix} g_{11} u^1 & g_{21} u^1 \end{bmatrix} + \begin{bmatrix} g_{12} u^2 & g_{22} u^2 \end{bmatrix} \right) \\ \left(\begin{bmatrix} g_{11} & g_{12} \end{bmatrix} u^1 + \begin{bmatrix} g_{21} & g_{22} \end{bmatrix} u^2 \right) & = \left(\begin{bmatrix} g_{11} u^1 & g_{12} u^1 \end{bmatrix} + \begin{bmatrix} g_{21} u^2 & g_{22} u^2 \end{bmatrix} \right) \end{cases} \begin{bmatrix} v^1 \\ v^2 \\ \hline v^1 \\ v^2 \end{bmatrix} \\
&= \begin{cases} \begin{bmatrix} g_{11} u^1 + g_{12} u^2 & g_{21} u^1 + g_{22} u^2 \end{bmatrix} & = \left(g_{11} u^1 + g_{12} u^2 \right) v^1 + \left(g_{21} u^1 + g_{22} u^2 \right) v^2 \\ \begin{bmatrix} g_{11} u^1 + g_{21} u^2 & g_{12} u^1 + g_{22} u^2 \end{bmatrix} & = \left(g_{11} u^1 + g_{21} u^2 \right) v^1 + \left(g_{12} u^1 + g_{22} u^2 \right) v^2 \end{cases} \quad \begin{bmatrix} v^1 \\ v^2 \\ \hline v^1 \\ v^2 \end{bmatrix} \\
&= \begin{cases} g_{11} u^1 v^1 + g_{12} u^2 v^1 + g_{21} u^1 v^2 + g_{22} u^2 v^2 & = g_{11} u^1 v^1 + g_{21} u^2 v^1 + g_{12} u^1 v^2 + g_{22} u^2 v^2 \\ & \text{right distribution needs } g_{ij} = g_{ji} \\ g_{11} u^1 v^1 + g_{21} u^2 v^1 + g_{12} u^1 v^2 + g_{22} u^2 v^2 & \text{left distribution seems to be more correct} \end{cases}
\end{aligned}
$$

$$g = \begin{bmatrix}\vec{e}_1\cdot & \vec{e}_2\cdot\end{bmatrix}\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}$$

$$\begin{aligned}
\vec{u}\cdot\vec{v} = g\left(\vec{u},\vec{v}\right) = g\vec{u}\vec{v} &= \begin{bmatrix}\vec{e}_1\cdot & \vec{e}_2\cdot\end{bmatrix}\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\begin{bmatrix}u^1\\u^2\end{bmatrix}\begin{bmatrix}v^1\\v^2\end{bmatrix}\\
&= \begin{cases}\begin{bmatrix}\vec{e}_1\cdot & \vec{e}_2\cdot\end{bmatrix}\begin{bmatrix}\overleftarrow{\epsilon}^1\\\overleftarrow{\epsilon}^2\end{bmatrix}\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\begin{bmatrix}\overleftarrow{\epsilon}^1\\\overleftarrow{\epsilon}^2\end{bmatrix}\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\begin{bmatrix}u^1\\u^2\end{bmatrix}\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\begin{bmatrix}v^1\\v^2\end{bmatrix}\\
\begin{bmatrix}\vec{e}_1\cdot & \vec{e}_2\cdot\end{bmatrix}\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\begin{bmatrix}\overleftarrow{\epsilon}^1\\\overleftarrow{\epsilon}^2\end{bmatrix}\begin{bmatrix}\overleftarrow{\epsilon}^1\\\overleftarrow{\epsilon}^2\end{bmatrix}\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\begin{bmatrix}u^1\\u^2\end{bmatrix}\begin{bmatrix}v^1\\v^2\end{bmatrix}\end{cases}\\
&= \begin{cases}\begin{bmatrix}\vec{e}_1\cdot & \vec{e}_2\cdot\end{bmatrix}1\cdot 1\begin{bmatrix}u^1\\u^2\end{bmatrix}\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\begin{bmatrix}v^1\\v^2\end{bmatrix} = \left(u^1\vec{e}_1\cdot + u^2\vec{e}_2\cdot\right)\left(\vec{e}_1v^1 + \vec{e}_2v^2\right)\\
\begin{bmatrix}\vec{e}_1\cdot & \vec{e}_2\cdot\end{bmatrix}\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\begin{bmatrix}\overleftarrow{\epsilon}^1\\\overleftarrow{\epsilon}^2\end{bmatrix}\cdot 1\cdot\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\begin{bmatrix}u^1\\u^2\end{bmatrix}\begin{bmatrix}v^1\\v^2\end{bmatrix}\end{cases}
\end{aligned}$$

$$\begin{aligned}
&g_{11}\overleftarrow{\epsilon}^1\overleftarrow{\epsilon}^1 + g_{12}\overleftarrow{\epsilon}^1\overleftarrow{\epsilon}^2 + g_{21}\overleftarrow{\epsilon}^2\overleftarrow{\epsilon}^1 + g_{22}\overleftarrow{\epsilon}^2\overleftarrow{\epsilon}^2 && = g_{ij}\overleftarrow{\epsilon}^i\overleftarrow{\epsilon}^j\\
=&g_{11}\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix} + g_{12}\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix} + g_{21}\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix} + g_{22}\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix} && \overleftarrow{\epsilon}^1 = \begin{bmatrix}1 & 0\end{bmatrix}\\
& && \overleftarrow{\epsilon}^2 = \begin{bmatrix}0 & 1\end{bmatrix}\\
=&g_{11}\begin{bmatrix}1\begin{bmatrix}1 & 0\end{bmatrix} & 0\begin{bmatrix}1 & 0\end{bmatrix}\end{bmatrix} + g_{12}\begin{bmatrix}1\begin{bmatrix}0 & 1\end{bmatrix} & 0\begin{bmatrix}0 & 1\end{bmatrix}\end{bmatrix}\\
+&g_{21}\begin{bmatrix}0\begin{bmatrix}1 & 0\end{bmatrix} & 1\begin{bmatrix}1 & 0\end{bmatrix}\end{bmatrix} + g_{22}\begin{bmatrix}0\begin{bmatrix}0 & 1\end{bmatrix} & 1\begin{bmatrix}0 & 1\end{bmatrix}\end{bmatrix} && \text{left distribution}\\
=&g_{11}\begin{bmatrix}\begin{bmatrix}1\cdot 1 & 1\cdot 0\end{bmatrix} & \begin{bmatrix}0\cdot 1 & 0\cdot 0\end{bmatrix}\end{bmatrix} + g_{12}\begin{bmatrix}\begin{bmatrix}1\cdot 0 & 1\cdot 1\end{bmatrix} & \begin{bmatrix}0\cdot 0 & 0\cdot 1\end{bmatrix}\end{bmatrix}\\
+&g_{21}\begin{bmatrix}\begin{bmatrix}0\cdot 1 & 0\cdot 0\end{bmatrix} & \begin{bmatrix}1\cdot 1 & 1\cdot 0\end{bmatrix}\end{bmatrix} + g_{22}\begin{bmatrix}\begin{bmatrix}0\cdot 0 & 0\cdot 1\end{bmatrix} & \begin{bmatrix}1\cdot 0 & 1\cdot 1\end{bmatrix}\end{bmatrix}\\
=&g_{11}\begin{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix} & \begin{bmatrix}0 & 0\end{bmatrix}\end{bmatrix} + g_{12}\begin{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix} & \begin{bmatrix}0 & 0\end{bmatrix}\end{bmatrix} + g_{21}\begin{bmatrix}\begin{bmatrix}0 & 0\end{bmatrix} & \begin{bmatrix}1 & 0\end{bmatrix}\end{bmatrix} + g_{22}\begin{bmatrix}\begin{bmatrix}0 & 0\end{bmatrix} & \begin{bmatrix}0 & 1\end{bmatrix}\end{bmatrix}\\
=&\begin{bmatrix}\begin{bmatrix}g_{11} & 0\end{bmatrix} & \begin{bmatrix}0 & 0\end{bmatrix}\end{bmatrix} + \begin{bmatrix}\begin{bmatrix}0 & g_{12}\end{bmatrix} & \begin{bmatrix}0 & 0\end{bmatrix}\end{bmatrix} + \begin{bmatrix}\begin{bmatrix}0 & 0\end{bmatrix} & \begin{bmatrix}g_{21} & 0\end{bmatrix}\end{bmatrix} + \begin{bmatrix}\begin{bmatrix}0 & 0\end{bmatrix} & \begin{bmatrix}0 & g_{22}\end{bmatrix}\end{bmatrix}\\
=&\begin{bmatrix}\begin{bmatrix}g_{11} & g_{12}\end{bmatrix} & \begin{bmatrix}g_{21} & g_{22}\end{bmatrix}\end{bmatrix} && = g\\
=&\begin{bmatrix}\begin{bmatrix}\vec{e}_1\cdot\vec{e}_1 & \vec{e}_1\cdot\vec{e}_2\end{bmatrix} & \begin{bmatrix}\vec{e}_2\cdot\vec{e}_1 & \vec{e}_2\cdot\vec{e}_2\end{bmatrix}\end{bmatrix} && g_{ij} = \vec{e}_i\cdot\vec{e}_j\\
=&\begin{bmatrix}\vec{e}_1\cdot\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix} & \vec{e}_2\cdot\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\end{bmatrix} && \text{left distribution}\\
=&\begin{bmatrix}\vec{e}_1\cdot & \vec{e}_2\cdot\end{bmatrix}\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}
\end{aligned}$$

$$g = g_{ij}\overleftarrow{\epsilon}^i\overleftarrow{\epsilon}^j$$

$$g = \widetilde{g}_{ij}\widetilde{\overleftarrow{\epsilon}}^i\widetilde{\overleftarrow{\epsilon}}^j$$

$(0,1)$-tensor
covariant
basis vector

$$\begin{cases}\widetilde{\vec{e}}_j = \vec{e}_i\widetilde{e}^i_j\\ \vec{e}_j = \widetilde{\vec{e}}_i e^i_j\end{cases}$$

covector component scalar

$$\begin{cases}\widetilde{\alpha}_j = \alpha_i\widetilde{e}^i_j\\ \alpha_j = \widetilde{\alpha}_i e^i_j\end{cases}$$

$(0,2)$-tensor
metric tensor

$$\begin{cases}\widetilde{g}_{ij} = g_{hk}\widetilde{e}^h_i\widetilde{e}^k_j\\ g_{ij} = \widetilde{g}_{hk}e^h_ie^k_j\end{cases}$$

$(1,0)$-tensor
contravariant
vector component scalar

$$\begin{cases}\widetilde{v}^i = e^i_jv^j\\ v^i = \widetilde{e}^i_j\widetilde{v}^j\end{cases}$$

dual basis covector

$$\begin{cases}\widetilde{\overleftarrow{\epsilon}}^i = e^i_j\overleftarrow{\epsilon}^j\\ \overleftarrow{\epsilon}^i = \widetilde{e}^i_j\widetilde{\overleftarrow{\epsilon}}^j\end{cases}$$

$(1,1)$-tensor

linear map

$$\begin{cases}\widetilde{L}^i_j = e^i_hL^h_k\widetilde{e}^k_j\\ L^i_j = \widetilde{e}^i_h\widetilde{L}^h_ke^k_j\end{cases}$$

$$\begin{aligned} g = g_{ij}\overleftarrow{\epsilon}^i\overleftarrow{\epsilon}^j &= g_{ij}\widetilde{e}^i_h\widetilde{\overleftarrow{\epsilon}}^h\widetilde{e}^j_k\widetilde{\overleftarrow{\epsilon}}^k \\ &= g_{ij}\widetilde{e}^i_h\widetilde{e}^j_k\widetilde{\overleftarrow{\epsilon}}^h\widetilde{\overleftarrow{\epsilon}}^k \end{aligned} \qquad \begin{aligned} \overleftarrow{\epsilon}^i &= \widetilde{e}^i_h\widetilde{\overleftarrow{\epsilon}}^h \\ \overleftarrow{\epsilon}^j &= \widetilde{e}^j_k\widetilde{\overleftarrow{\epsilon}}^k \end{aligned} \Leftarrow \overleftarrow{\epsilon}^i = \widetilde{e}^i_j\widetilde{\overleftarrow{\epsilon}}^j$$

$$g = \widetilde{g}_{ij}\widetilde{\overleftarrow{\epsilon}}^i\widetilde{\overleftarrow{\epsilon}}^j = g_{ij}\widetilde{e}^i_h\widetilde{e}^j_k\widetilde{\overleftarrow{\epsilon}}^h\widetilde{\overleftarrow{\epsilon}}^k = g_{hk}\widetilde{e}^h_i\widetilde{e}^k_j\widetilde{\overleftarrow{\epsilon}}^i\widetilde{\overleftarrow{\epsilon}}^j \qquad \text{RHS}: \begin{matrix} h \to i \\ i \to h \\ j \to k \\ k \to j \end{matrix}$$

$$\begin{aligned} \widetilde{g}_{ij}\cancel{\widetilde{\overleftarrow{\epsilon}}^i}\cancel{\widetilde{\overleftarrow{\epsilon}}^j} &= g_{hk}\widetilde{e}^h_i\widetilde{e}^k_j\cancel{\widetilde{\overleftarrow{\epsilon}}^i}\cancel{\widetilde{\overleftarrow{\epsilon}}^j} \\ \widetilde{g}_{ij} &= g_{hk}\widetilde{e}^h_i\widetilde{e}^k_j \\ &\Downarrow \\ g_{ij} &= \widetilde{g}_{hk}e^h_i e^k_j \end{aligned}$$

matrix multiplication

$$\begin{bmatrix} \overleftarrow{\epsilon}^1 \\ \overleftarrow{\epsilon}^2 \end{bmatrix} \begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix} = \begin{bmatrix} \overleftarrow{\epsilon}^1\vec{e}_1 & \overleftarrow{\epsilon}^1\vec{e}_2 \\ \overleftarrow{\epsilon}^2\vec{e}_1 & \overleftarrow{\epsilon}^2\vec{e}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{aligned} \vec{v}\overleftarrow{\alpha} = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix}\begin{bmatrix} v^1 \\ v^2 \end{bmatrix}\begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix}\begin{bmatrix} \overleftarrow{\epsilon}^1 \\ \overleftarrow{\epsilon}^2 \end{bmatrix} &= \begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix}\begin{bmatrix} v^1\alpha_1 & v^1\alpha_2 \\ v^2\alpha_1 & v^2\alpha_2 \end{bmatrix}\begin{bmatrix} \overleftarrow{\epsilon}^1 \\ \overleftarrow{\epsilon}^2 \end{bmatrix} \\ &= \vec{e}_1 v^1\alpha_1\overleftarrow{\epsilon}^1 + \vec{e}_1 v^1\alpha_2\overleftarrow{\epsilon}^2 + \vec{e}_2 v^2\alpha_1\overleftarrow{\epsilon}^1 + \vec{e}_2 v^2\alpha_2\overleftarrow{\epsilon}^2 \\ &= v^1\alpha_1\vec{e}_1\overleftarrow{\epsilon}^1 + v^1\alpha_2\vec{e}_1\overleftarrow{\epsilon}^2 + v^2\alpha_1\vec{e}_2\overleftarrow{\epsilon}^1 + v^2\alpha_2\vec{e}_2\overleftarrow{\epsilon}^2 \end{aligned}$$

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \overleftarrow{\epsilon}^1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad \overleftarrow{\epsilon}^2 = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$= v^1\alpha_1\begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix} + v^1\alpha_2\begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix} + v^2\alpha_1\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix} + v^2\alpha_2\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$= v^1\alpha_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + v^1\alpha_2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + v^2\alpha_1\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + v^2\alpha_2\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} v^1\alpha_1 & v^1\alpha_2 \\ v^2\alpha_1 & v^2\alpha_2 \end{bmatrix}$$

$$\begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix}\begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \alpha_1 v^1 + \alpha_2 v^2$$

$$\begin{bmatrix} L^1_1 & L^1_2 \\ L^2_1 & L^2_2 \end{bmatrix}\begin{bmatrix} M^1_1 & M^1_2 \\ M^2_1 & M^2_2 \end{bmatrix} = \begin{bmatrix} L^1_1 M^1_1 + L^1_2 M^2_1 & L^1_1 M^1_2 + L^1_2 M^2_2 \\ L^2_1 M^1_1 + L^2_2 M^2_1 & L^2_1 M^1_2 + L^2_2 M^2_2 \end{bmatrix}$$

Kronecker product

$$\begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \otimes \begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix} = \begin{cases} \begin{bmatrix} v^1\begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix} \\ v^2\begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} v^1\alpha_1 & v^1\alpha_2 \end{bmatrix} \\ \begin{bmatrix} v^2\alpha_1 & v^2\alpha_2 \end{bmatrix} \end{bmatrix} & \text{left distribution} \\ \begin{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix}\alpha_1 & \begin{bmatrix} v^1 \\ v^2 \end{bmatrix}\alpha_2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} v^1\alpha_1 \\ v^2\alpha_1 \end{bmatrix} & \begin{bmatrix} v^1\alpha_2 \\ v^2\alpha_2 \end{bmatrix} \end{bmatrix} & \text{right distribution} \end{cases}$$

$$\begin{bmatrix} \overleftarrow{\epsilon}^1 \\ \overleftarrow{\epsilon}^2 \end{bmatrix} \otimes \begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix} = \begin{cases} \begin{cases} \begin{bmatrix} \overleftarrow{\epsilon}^1\begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix} \\ \overleftarrow{\epsilon}^2\begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \overleftarrow{\epsilon}^1\vec{e}_1 & \overleftarrow{\epsilon}^1\vec{e}_2 \end{bmatrix} \\ \begin{bmatrix} \overleftarrow{\epsilon}^2\vec{e}_1 & \overleftarrow{\epsilon}^2\vec{e}_2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 1 \end{bmatrix} \end{bmatrix} & \text{left distribution} \\ \begin{bmatrix} \begin{bmatrix} \overleftarrow{\epsilon}^1 \\ \overleftarrow{\epsilon}^2 \end{bmatrix}\vec{e}_1 & \begin{bmatrix} \overleftarrow{\epsilon}^1 \\ \overleftarrow{\epsilon}^2 \end{bmatrix}\vec{e}_2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \overleftarrow{\epsilon}^1\vec{e}_1 \\ \overleftarrow{\epsilon}^2\vec{e}_1 \end{bmatrix} & \begin{bmatrix} \overleftarrow{\epsilon}^1\vec{e}_2 \\ \overleftarrow{\epsilon}^2\vec{e}_2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} & \text{right distribution} \end{cases} & \otimes \text{ not preserved} \\ \begin{cases} \begin{bmatrix} \overleftarrow{\epsilon}^1 \otimes \begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix} \\ \overleftarrow{\epsilon}^2 \otimes \begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \overleftarrow{\epsilon}^1 \otimes \vec{e}_1 & \overleftarrow{\epsilon}^1 \otimes \vec{e}_2 \end{bmatrix} \\ \begin{bmatrix} \overleftarrow{\epsilon}^2 \otimes \vec{e}_1 & \overleftarrow{\epsilon}^2 \otimes \vec{e}_2 \end{bmatrix} \end{bmatrix} & \text{left distribution} \\ \begin{bmatrix} \begin{bmatrix} \overleftarrow{\epsilon}^1 \\ \overleftarrow{\epsilon}^2 \end{bmatrix} \otimes \vec{e}_1 & \begin{bmatrix} \overleftarrow{\epsilon}^1 \\ \overleftarrow{\epsilon}^2 \end{bmatrix} \otimes \vec{e}_2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \overleftarrow{\epsilon}^1 \otimes \vec{e}_1 \\ \overleftarrow{\epsilon}^2 \otimes \vec{e}_1 \end{bmatrix} & \begin{bmatrix} \overleftarrow{\epsilon}^1 \otimes \vec{e}_2 \\ \overleftarrow{\epsilon}^2 \otimes \vec{e}_2 \end{bmatrix} \end{bmatrix} & \text{right distribution} \end{cases} & \otimes \text{ preserved} \end{cases}$$

$$
\begin{bmatrix}\alpha_1 & \alpha_2\end{bmatrix} \otimes \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \begin{cases}
\begin{bmatrix} \alpha_1 \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} & \alpha_2 \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \alpha_1 v^1 \\ \alpha_1 v^2 \end{bmatrix} & \begin{bmatrix} \alpha_2 v^1 \\ \alpha_2 v^2 \end{bmatrix} \end{bmatrix} & \text{left distribution} \\
\begin{bmatrix} \begin{bmatrix}\alpha_1 & \alpha_2\end{bmatrix} v^1 \\ \begin{bmatrix}\alpha_1 & \alpha_2\end{bmatrix} v^2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \alpha_1 v^1 & \alpha_2 v^1 \end{bmatrix} \\ \begin{bmatrix} \alpha_1 v^2 & \alpha_2 v^2 \end{bmatrix} \end{bmatrix} & \text{right distribution}
\end{cases}
$$

$$
\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix} \otimes \begin{bmatrix} \overleftarrow{\epsilon}^1 \\ \overleftarrow{\epsilon}^2 \end{bmatrix} = \begin{cases}
\begin{cases}
\begin{bmatrix} \vec{e}_1 \begin{bmatrix} \overleftarrow{\epsilon}^1 \\ \overleftarrow{\epsilon}^2 \end{bmatrix} & \vec{e}_2 \begin{bmatrix} \overleftarrow{\epsilon}^1 \\ \overleftarrow{\epsilon}^2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \vec{e}_1 \overleftarrow{\epsilon}^1 \\ \vec{e}_1 \overleftarrow{\epsilon}^2 \end{bmatrix} & \begin{bmatrix} \vec{e}_2 \overleftarrow{\epsilon}^1 \\ \vec{e}_2 \overleftarrow{\epsilon}^2 \end{bmatrix} \end{bmatrix} & \text{left distribution} \\
\begin{bmatrix} \begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix} \overleftarrow{\epsilon}^1 \\ \begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix} \overleftarrow{\epsilon}^2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \vec{e}_1 \overleftarrow{\epsilon}^1 & \vec{e}_2 \overleftarrow{\epsilon}^1 \end{bmatrix} \\ \begin{bmatrix} \vec{e}_1 \overleftarrow{\epsilon}^2 & \vec{e}_2 \overleftarrow{\epsilon}^2 \end{bmatrix} \end{bmatrix} & \text{right distribution}
\end{cases} & \otimes \text{ not preserved} \\
\begin{cases}
\begin{bmatrix} \vec{e}_1 \otimes \begin{bmatrix} \overleftarrow{\epsilon}^1 \\ \overleftarrow{\epsilon}^2 \end{bmatrix} & \vec{e}_2 \otimes \begin{bmatrix} \overleftarrow{\epsilon}^1 \\ \overleftarrow{\epsilon}^2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \vec{e}_1 \otimes \overleftarrow{\epsilon}^1 \\ \vec{e}_1 \otimes \overleftarrow{\epsilon}^2 \end{bmatrix} & \begin{bmatrix} \vec{e}_2 \otimes \overleftarrow{\epsilon}^1 \\ \vec{e}_2 \otimes \overleftarrow{\epsilon}^2 \end{bmatrix} \end{bmatrix} & \text{left distribution} \\
\begin{bmatrix} \begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix} \otimes \overleftarrow{\epsilon}^1 \\ \begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix} \otimes \overleftarrow{\epsilon}^2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \vec{e}_1 \otimes \overleftarrow{\epsilon}^1 & \vec{e}_2 \otimes \overleftarrow{\epsilon}^1 \end{bmatrix} \\ \begin{bmatrix} \vec{e}_1 \otimes \overleftarrow{\epsilon}^2 & \vec{e}_2 \otimes \overleftarrow{\epsilon}^2 \end{bmatrix} \end{bmatrix} & \text{right distribution}
\end{cases} & \otimes \text{ preserved}
\end{cases}
$$

$$
\begin{bmatrix} u^1 \\ u^2 \end{bmatrix} \otimes \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \begin{cases}
\begin{bmatrix} u^1 \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \\ u^2 \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} u^1 v^1 \\ u^1 v^2 \end{bmatrix} \\ \begin{bmatrix} u^2 v^1 \\ u^2 v^2 \end{bmatrix} \end{bmatrix} & \text{left distribution} \\
\begin{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} v^1 \\ \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} v^2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} u^1 v^1 \\ u^2 v^1 \end{bmatrix} \\ \begin{bmatrix} u^1 v^2 \\ u^2 v^2 \end{bmatrix} \end{bmatrix} & \text{right distribution}
\end{cases}
$$

$$
\begin{bmatrix} \overleftarrow{\epsilon}^1 \\ \overleftarrow{\epsilon}^2 \end{bmatrix} \otimes \begin{bmatrix} \overleftarrow{\epsilon}^1 \\ \overleftarrow{\epsilon}^2 \end{bmatrix} = \begin{cases}
\begin{cases}
\begin{bmatrix} \overleftarrow{\epsilon}^1 \begin{bmatrix} \overleftarrow{\epsilon}^1 \\ \overleftarrow{\epsilon}^2 \end{bmatrix} \\ \overleftarrow{\epsilon}^2 \begin{bmatrix} \overleftarrow{\epsilon}^1 \\ \overleftarrow{\epsilon}^2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \overleftarrow{\epsilon}^1 \overleftarrow{\epsilon}^1 \\ \overleftarrow{\epsilon}^1 \overleftarrow{\epsilon}^2 \end{bmatrix} \\ \begin{bmatrix} \overleftarrow{\epsilon}^2 \overleftarrow{\epsilon}^1 \\ \overleftarrow{\epsilon}^2 \overleftarrow{\epsilon}^2 \end{bmatrix} \end{bmatrix} & \text{left distribution} \\
\begin{bmatrix} \begin{bmatrix} \overleftarrow{\epsilon}^1 \\ \overleftarrow{\epsilon}^2 \end{bmatrix} \overleftarrow{\epsilon}^1 \\ \begin{bmatrix} \overleftarrow{\epsilon}^1 \\ \overleftarrow{\epsilon}^2 \end{bmatrix} \overleftarrow{\epsilon}^2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \overleftarrow{\epsilon}^1 \overleftarrow{\epsilon}^1 \\ \overleftarrow{\epsilon}^2 \overleftarrow{\epsilon}^1 \end{bmatrix} \\ \begin{bmatrix} \overleftarrow{\epsilon}^1 \overleftarrow{\epsilon}^2 \\ \overleftarrow{\epsilon}^2 \overleftarrow{\epsilon}^2 \end{bmatrix} \end{bmatrix} & \text{right distribution}
\end{cases} & \otimes \text{ not preserved} \\
\begin{cases}
\begin{bmatrix} \overleftarrow{\epsilon}^1 \otimes \begin{bmatrix} \overleftarrow{\epsilon}^1 \\ \overleftarrow{\epsilon}^2 \end{bmatrix} \\ \overleftarrow{\epsilon}^2 \otimes \begin{bmatrix} \overleftarrow{\epsilon}^1 \\ \overleftarrow{\epsilon}^2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \overleftarrow{\epsilon}^1 \otimes \overleftarrow{\epsilon}^1 \\ \overleftarrow{\epsilon}^1 \otimes \overleftarrow{\epsilon}^2 \end{bmatrix} \\ \begin{bmatrix} \overleftarrow{\epsilon}^2 \otimes \overleftarrow{\epsilon}^1 \\ \overleftarrow{\epsilon}^2 \otimes \overleftarrow{\epsilon}^2 \end{bmatrix} \end{bmatrix} & \text{left distribution} \\
\begin{bmatrix} \begin{bmatrix} \overleftarrow{\epsilon}^1 \\ \overleftarrow{\epsilon}^2 \end{bmatrix} \otimes \overleftarrow{\epsilon}^1 \\ \begin{bmatrix} \overleftarrow{\epsilon}^1 \\ \overleftarrow{\epsilon}^2 \end{bmatrix} \otimes \overleftarrow{\epsilon}^2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \overleftarrow{\epsilon}^1 \otimes \overleftarrow{\epsilon}^1 \\ \overleftarrow{\epsilon}^2 \otimes \overleftarrow{\epsilon}^1 \end{bmatrix} \\ \begin{bmatrix} \overleftarrow{\epsilon}^1 \otimes \overleftarrow{\epsilon}^2 \\ \overleftarrow{\epsilon}^2 \otimes \overleftarrow{\epsilon}^2 \end{bmatrix} \end{bmatrix} & \text{right distribution}
\end{cases} & \otimes \text{ preserved}
\end{cases}
$$

$$
\begin{bmatrix}\alpha_1 & \alpha_2\end{bmatrix} \otimes \begin{bmatrix}\beta_1 & \beta_2\end{bmatrix} = \begin{cases}
\begin{bmatrix} \alpha_1 \begin{bmatrix}\beta_1 & \beta_2\end{bmatrix} & \alpha_2 \begin{bmatrix}\beta_1 & \beta_2\end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix}\alpha_1\beta_1 & \alpha_1\beta_2\end{bmatrix} & \begin{bmatrix}\alpha_2\beta_1 & \alpha_2\beta_2\end{bmatrix} \end{bmatrix} & \text{left distribution} \\
\begin{bmatrix} \begin{bmatrix}\alpha_1 & \alpha_2\end{bmatrix} \beta_1 & \begin{bmatrix}\alpha_1 & \alpha_2\end{bmatrix} \beta_2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix}\alpha_1\beta_1 & \alpha_2\beta_1\end{bmatrix} & \begin{bmatrix}\alpha_1\beta_2 & \alpha_2\beta_2\end{bmatrix} \end{bmatrix} & \text{right distribution}
\end{cases}
$$

$$
\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix} \otimes \begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix} = \begin{cases}
\begin{cases}
\begin{bmatrix} \vec{e}_1 \begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix} & \vec{e}_2 \begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix}\vec{e}_1\vec{e}_1 & \vec{e}_1\vec{e}_2\end{bmatrix} & \begin{bmatrix}\vec{e}_2\vec{e}_1 & \vec{e}_2\vec{e}_2\end{bmatrix} \end{bmatrix} & \text{left distribution} \\
\begin{bmatrix} \begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix} \vec{e}_1 & \begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix} \vec{e}_2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix}\vec{e}_1\vec{e}_1 & \vec{e}_2\vec{e}_1\end{bmatrix} & \begin{bmatrix}\vec{e}_1\vec{e}_2 & \vec{e}_2\vec{e}_2\end{bmatrix} \end{bmatrix} & \text{right distribution}
\end{cases} & \otimes \text{ not preserved} \\
\begin{cases}
\begin{bmatrix} \vec{e}_1 \otimes \begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix} & \vec{e}_2 \otimes \begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix}\vec{e}_1 \otimes \vec{e}_1 & \vec{e}_1 \otimes \vec{e}_2\end{bmatrix} & \begin{bmatrix}\vec{e}_2 \otimes \vec{e}_1 & \vec{e}_2 \otimes \vec{e}_2\end{bmatrix} \end{bmatrix} & \text{left distribution} \\
\begin{bmatrix} \begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix} \otimes \vec{e}_1 & \begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix} \otimes \vec{e}_2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix}\vec{e}_1 \otimes \vec{e}_1 & \vec{e}_2 \otimes \vec{e}_1\end{bmatrix} & \begin{bmatrix}\vec{e}_1 \otimes \vec{e}_2 & \vec{e}_2 \otimes \vec{e}_2\end{bmatrix} \end{bmatrix} & \text{right distribution}
\end{cases} & \otimes \text{ preserved}
\end{cases}
$$

$$\begin{bmatrix} L^1_1 & L^1_2 \\ L^2_1 & L^2_2 \end{bmatrix} \otimes \begin{bmatrix} M^1_1 & M^1_2 \\ M^2_1 & M^2_2 \end{bmatrix} = \begin{cases} \begin{bmatrix} L^1_1 \begin{bmatrix} M^1_1 & M^1_2 \\ M^2_1 & M^2_2 \end{bmatrix} & L^1_2 \begin{bmatrix} M^1_1 & M^1_2 \\ M^2_1 & M^2_2 \end{bmatrix} \\ L^2_1 \begin{bmatrix} M^1_1 & M^1_2 \\ M^2_1 & M^2_2 \end{bmatrix} & L^2_2 \begin{bmatrix} M^1_1 & M^1_2 \\ M^2_1 & M^2_2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} L^1_1 M^1_1 & L^1_1 M^1_2 \\ L^1_1 M^2_1 & L^1_1 M^2_2 \\ L^2_1 M^1_1 & L^2_1 M^1_2 \\ L^2_1 M^2_1 & L^2_1 M^2_2 \end{bmatrix} & \begin{bmatrix} L^1_2 M^1_1 & L^1_2 M^1_2 \\ L^1_2 M^2_1 & L^1_2 M^2_2 \\ L^2_2 M^1_1 & L^2_2 M^1_2 \\ L^2_2 M^2_1 & L^2_2 M^2_2 \end{bmatrix} \end{bmatrix} & \text{left distribution} \\ \begin{bmatrix} \begin{bmatrix} L^1_1 & L^1_2 \\ L^2_1 & L^2_2 \end{bmatrix} M^1_1 & \begin{bmatrix} L^1_1 & L^1_2 \\ L^2_1 & L^2_2 \end{bmatrix} M^1_2 \\ \begin{bmatrix} L^1_1 & L^1_2 \\ L^2_1 & L^2_2 \end{bmatrix} M^2_1 & \begin{bmatrix} L^1_1 & L^1_2 \\ L^2_1 & L^2_2 \end{bmatrix} M^2_2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} L^1_1 M^1_1 & L^1_2 M^1_1 \\ L^2_1 M^1_1 & L^2_2 M^1_1 \\ L^1_1 M^2_1 & L^1_2 M^2_1 \\ L^2_1 M^2_1 & L^2_2 M^2_1 \end{bmatrix} & \begin{bmatrix} L^1_1 M^1_2 & L^1_2 M^1_2 \\ L^2_1 M^1_2 & L^2_2 M^1_2 \\ L^1_1 M^2_2 & L^1_2 M^2_2 \\ L^2_1 M^2_2 & L^2_2 M^2_2 \end{bmatrix} \end{bmatrix} & \text{right distribution} \end{cases}$$

$$\begin{bmatrix} \begin{bmatrix} L^1_1 M^1_1 & L^1_1 M^1_2 \\ L^1_1 M^2_1 & L^1_1 M^2_2 \\ L^2_1 M^1_1 & L^2_1 M^1_2 \\ L^2_1 M^2_1 & L^2_1 M^2_2 \end{bmatrix} & \begin{bmatrix} L^1_2 M^1_1 & L^1_2 M^1_2 \\ L^1_2 M^2_1 & L^1_2 M^2_2 \\ L^2_2 M^1_1 & L^2_2 M^1_2 \\ L^2_2 M^2_1 & L^2_2 M^2_2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} N^{11}_{11} & N^{11}_{12} \\ N^{12}_{11} & N^{12}_{12} \\ N^{21}_{11} & N^{21}_{12} \\ N^{22}_{11} & N^{22}_{12} \end{bmatrix} & \begin{bmatrix} N^{11}_{21} & N^{11}_{22} \\ N^{12}_{21} & N^{12}_{22} \\ N^{21}_{21} & N^{21}_{22} \\ N^{22}_{21} & N^{22}_{22} \end{bmatrix} \end{bmatrix} \overset{\begin{cases} 11 \to 1 \\ 12 \to 2 \\ 21 \to 3 \\ 22 \to 4 \end{cases}}{=} \begin{bmatrix} \begin{bmatrix} N^1_1 & N^1_2 \\ N^2_1 & N^2_2 \\ N^3_1 & N^3_2 \\ N^4_1 & N^4_2 \end{bmatrix} & \begin{bmatrix} N^1_3 & N^1_4 \\ N^2_3 & N^2_4 \\ N^3_3 & N^3_4 \\ N^4_3 & N^4_4 \end{bmatrix} \end{bmatrix}$$

Kronecker product Shinonome Masaki notation 東雲正樹

$$(A \otimes B)(C \otimes D) = EF$$

$$[(A \otimes B)(C \otimes D)]^{ij}{}_{uv} = (EF)^{ij}{}_{uv} = E^{ij}{}_{hk} F^{hk}{}_{uv} = (A \otimes B)^{ij}{}_{hk} (C \otimes D)^{hk}{}_{uv} = A^i{}_h B^j{}_k C^h{}_u D^k{}_v$$

tensor product with index notation

$$\begin{aligned} \mathcal{V} \otimes \mathcal{V}^* \ni \vec{v} \otimes \overleftarrow{\alpha} &= \vec{e}_i v^i \otimes \alpha_j \overleftarrow{\epsilon}^j & &= \vec{e}_i v^i \alpha_j \overleftarrow{\epsilon}^j = \vec{v}\overleftarrow{\alpha} \\ &= v^i \vec{e}_i \otimes \alpha_j \overleftarrow{\epsilon}^j & &= v^i \vec{e}_i \alpha_j \overleftarrow{\epsilon}^j \\ &= v^i \alpha_j \vec{e}_i \otimes \overleftarrow{\epsilon}^j & &= v^i \alpha_j \vec{e}_i \overleftarrow{\epsilon}^j \end{aligned}$$

pure (1,1)-tensor $\vec{e}_i L^i_j \overleftarrow{\epsilon}^j = \vec{e}_i v^i \alpha_j \overleftarrow{\epsilon}^j$; impure (1,1)-tensor $\vec{e}_i L^i_j \overleftarrow{\epsilon}^j \not\equiv \vec{e}_i v^i \alpha_j \overleftarrow{\epsilon}^j$

$$\begin{aligned} \overset{\rightarrow\leftarrow}{L} = L &= \vec{e}_i L^i_j \overleftarrow{\epsilon}^j & &\overset{\text{e.g.}}{=} \begin{bmatrix} L^1_1 & L^1_2 \\ L^2_1 & L^2_2 \end{bmatrix} \\ &= L^i_j \vec{e}_i \overleftarrow{\epsilon}^j & &= L^i_j \vec{e}_i \otimes \overleftarrow{\epsilon}^j \in \mathcal{V} \otimes \mathcal{V}^* \end{aligned}$$

$$\begin{aligned} g &= g_{ij} \overleftarrow{\epsilon}^i \overleftarrow{\epsilon}^j & &\overset{\text{e.g.}}{=} \begin{bmatrix} \begin{bmatrix} g_{11} & g_{12} \end{bmatrix} & \begin{bmatrix} g_{21} & g_{22} \end{bmatrix} \end{bmatrix} \\ &= g_{ij} \overleftarrow{\epsilon}^i \otimes \overleftarrow{\epsilon}^j & &\in \mathcal{V}^* \otimes \mathcal{V}^* \end{aligned}$$

bilinear form = 2-form

$$b : \mathcal{V} \times \mathcal{V} \to \mathbb{F} \Leftrightarrow \mathcal{V}^2 \xrightarrow{b} \mathbb{F} \Leftrightarrow b \in \mathbb{F}^{\mathcal{V}^2}$$

$$\begin{aligned} b &= b_{ij} \overleftarrow{\epsilon}^i \overleftarrow{\epsilon}^j & &\overset{\text{e.g.}}{=} \begin{bmatrix} \begin{bmatrix} b_{11} & b_{12} \end{bmatrix} & \begin{bmatrix} b_{21} & b_{22} \end{bmatrix} \end{bmatrix} \\ &= b_{ij} \overleftarrow{\epsilon}^i \otimes \overleftarrow{\epsilon}^j & &\in \mathcal{V}^* \otimes \mathcal{V}^* \end{aligned}$$

mathematically,

inner product	norm	metric / distance function	
$\langle \cdot \vert \cdot \rangle = \langle \square \vert \square \rangle : \mathcal{V}^2 \xrightarrow{g} \mathbb{F}$	$\Vert \cdot \Vert = \Vert \square \Vert : \mathcal{V} \to \mathbb{R}$	$d(\cdot, \cdot) = d(\square, \square) : X^2 \to \mathbb{R}$	function (f)
$\left\langle \vec{v} \vert \vec{v} \right\rangle \geq 0 \quad , \left\langle \vec{v} \vert \vec{v} \right\rangle = 0 \Leftrightarrow \vec{v} = \vec{0}$	$\Vert \vec{v} \Vert \geq 0, \quad \Vert \vec{v} \Vert = 0 \Leftrightarrow \vec{v} = \vec{0}$	$d(x, y) \geq 0, \quad d(x, y) = 0 \Leftrightarrow x = y$	positivity (p)
$\left\langle \vec{u} \vert \vec{v} \right\rangle = \left\langle \vec{v} \vert \vec{u} \right\rangle^*$		$d(x, y) = d(y, x)$	symmetry(s)
$\left\langle \vec{u} \vert s\vec{v} \right\rangle = s \left\langle \vec{u} \vert \vec{v} \right\rangle$	$\Vert s\vec{v} \Vert = \vert s \vert \, \Vert \vec{v} \Vert$		homogeneity (h)
$\left\langle \vec{u} \vert \vec{v} + \vec{w} \right\rangle = \left\langle \vec{u} \vert \vec{v} \right\rangle + \left\langle \vec{u} \vert \vec{w} \right\rangle$	$\Vert \vec{u} + \vec{v} \Vert \leq \Vert \vec{u} \Vert + \Vert \vec{v} \Vert$	$d(x, y) \leq d(x, z) + d(z, y)$	additivity (a)

[2]

$$
\begin{aligned}
& g \text{ is a metric tensor} \\
\Leftrightarrow & \begin{cases} \text{metric tensor is a bilinear form or 2-form} \quad \text{linearity}\,(l) \begin{cases}(h)\\(a)\end{cases} \\ \text{positivity}\,(p) \\ \text{symmetry}(s) \end{cases} \\
\Leftrightarrow & \begin{cases} g \in \mathbb{F}^{\mathcal{V}^2} & \Leftrightarrow g : \mathcal{V} \times \mathcal{V} \to \mathbb{F} \\ g\left(\vec{u}, \vec{v}\right) \geq 0 & \text{positivity}\,(p) \\ g\left(\vec{u}, \vec{v}\right) = g\left(\vec{v}, \vec{u}\right)^* & \text{conjugate symmetry}(s) \\ g\left(\vec{u}, s\vec{v}\right) = sg\left(\vec{u}, \vec{v}\right) & \text{homogeneity}\,(h) \\ g\left(\vec{u}, \vec{v} + \vec{w}\right) = g\left(\vec{u}, \vec{v}\right) + g\left(\vec{u}, \vec{w}\right) & \text{additivity}\,(a) \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
&\begin{bmatrix}\begin{bmatrix} g_{11} & g_{12} \end{bmatrix} & \begin{bmatrix} g_{21} & g_{22} \end{bmatrix}\end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \equiv \begin{bmatrix} u^1 & u^2 \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \\
&\begin{bmatrix} u^1 & u^2 \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \left(\begin{bmatrix} v^1 \\ v^2 \end{bmatrix} + \begin{bmatrix} w^1 \\ w^2 \end{bmatrix} \right) = \begin{bmatrix} u^1 & u^2 \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} + \begin{bmatrix} u^1 & u^2 \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} w^1 \\ w^2 \end{bmatrix} && g\left(\vec{u}, \vec{v} + \vec{w}\right) = g\left(\vec{u}, \vec{v}\right) + g\left(\vec{u}, \vec{w}\right) \\
&\left(\begin{bmatrix} t^1 & t^2 \end{bmatrix} + \begin{bmatrix} u^1 & u^2 \end{bmatrix} \right) \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \begin{bmatrix} t^1 & t^2 \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} + \begin{bmatrix} u^1 & u^2 \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} && g\left(\vec{t} + \vec{u}, \vec{v}\right) = g\left(\vec{t}, \vec{v}\right) + g\left(\vec{u}, \vec{v}\right) \\
&\begin{bmatrix} s\,u^1 & s\,u^2 \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = s \begin{bmatrix} u^1 & u^2 \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \begin{bmatrix} u^1 & u^2 \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} sv^1 \\ sv^2 \end{bmatrix} && g\left(s\vec{u}, \vec{v}\right) = sg\left(\vec{u}, \vec{v}\right) = g\left(\vec{u}, s\vec{v}\right) \\
&= \begin{bmatrix} u^1 & u^2 \end{bmatrix} \begin{bmatrix} s\,g_{11} & s\,g_{12} \\ s\,g_{21} & s\,g_{22} \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \neq \begin{bmatrix} s\,u^1 & s\,u^2 \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} s\,v^1 \\ s\,v^2 \end{bmatrix} = s^2 \begin{bmatrix} u^1 & u^2 \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} && \neq g\left(s\vec{u}, s\vec{v}\right) = s^2 g\left(\vec{u}, \vec{v}\right)
\end{aligned}
$$

$$
\begin{aligned}
& g \text{ is a metric tensor} \\
\Leftrightarrow & \begin{cases} \text{metric tensor is a bilinear form or 2-form} \quad \text{linearity}\,(l) \begin{cases}(h)\\(a)\end{cases} \\ \text{positivity}\,(p) \\ \text{symmetry}(s) \end{cases} \\
\Leftrightarrow & \begin{cases} g \in \mathbb{F}^{\mathcal{V}^2} & \Leftrightarrow g : \mathcal{V} \times \mathcal{V} \to \mathbb{F} \\ g\left(\vec{u}, \vec{v}\right) \geq 0 & \text{positivity}\,(p) \\ g\left(\vec{u}, \vec{v}\right) = g\left(\vec{v}, \vec{u}\right)^* & \text{conjugate symmetry}(s) \\ g\left(\vec{u}, s\vec{v}\right) = s\,g\left(\vec{u}, \vec{v}\right) & \text{homogeneity}\,(h) \\ g\left(\vec{u}, \vec{v} + \vec{w}\right) = g\left(\vec{u}, \vec{v}\right) + g\left(\vec{u}, \vec{w}\right) & \text{additivity}\,(a) \end{cases} \\
\Rightarrow & \begin{cases} g \in \mathbb{R}^{\mathcal{V}^2} & \Leftrightarrow g : \mathcal{V} \times \mathcal{V} \to \mathbb{R} \\ g\left(\vec{u}, \vec{v}\right) \geq 0 & \text{positivity}\,(p) \\ g\left(\vec{u}, \vec{v}\right) = g\left(\vec{v}, \vec{u}\right) & \text{symmetry}(s) \\ g\left(s\vec{u}, \vec{v}\right) = s\,g\left(\vec{u}, \vec{v}\right) = g\left(\vec{u}, s\vec{v}\right) & \text{homogeneity}\,(h) \\ g\left(\vec{t} + \vec{u}, \vec{v}\right) = g\left(\vec{t}, \vec{v}\right) + g\left(\vec{u}, \vec{v}\right) & \text{additivity}\,(a1) \\ g\left(\vec{u}, \vec{v} + \vec{w}\right) = g\left(\vec{u}, \vec{v}\right) + g\left(\vec{u}, \vec{w}\right) & \text{additivity}\,(a2) \end{cases} \\
\Rightarrow & \begin{cases} g \in \mathbb{R}_{\geq 0}^{\mathcal{V}^2} & \Leftrightarrow g : \mathcal{V} \times \mathcal{V} \to \mathbb{R}_{\geq 0} \\ g\left(\vec{u}, \vec{v}\right) = g\left(\vec{v}, \vec{u}\right) & \text{symmetry}(s) \\ g\left(s\vec{u}, \vec{v}\right) = s\,g\left(\vec{u}, \vec{v}\right) = g\left(\vec{u}, s\vec{v}\right) & \text{homogeneity}\,(h) \\ g\left(\vec{t} + \vec{u}, \vec{v}\right) = g\left(\vec{t}, \vec{v}\right) + g\left(\vec{u}, \vec{v}\right) & \text{additivity}\,(a1) \\ g\left(\vec{u}, \vec{v} + \vec{w}\right) = g\left(\vec{u}, \vec{v}\right) + g\left(\vec{u}, \vec{w}\right) & \text{additivity}\,(a2) \end{cases}
\end{aligned}
$$

[2] metric/distance: positive definiteness $d(x, y) \geq 0$ 一定有嗎? 一個顯然的例子就是, 在相對論時空下, 我們取了洛倫茲度規, 此時的向量就會根據其平方值大於零, 等於零, 小於零三種情況分爲類時, 類光, 類空三種向量(洛倫茲度規也分兩種等價的號差, 所以這仨向量名稱的用法並不統一).

b is a bilinear form or 2-form

$$\Leftrightarrow \begin{cases} b \in \mathbb{F}^{\mathcal{V}^2} & \Leftrightarrow b : \mathcal{V} \times \mathcal{V} \to \mathbb{F} \\ b\left(s\vec{u}, \vec{v}\right) = s\, b\left(\vec{u}, \vec{v}\right) & \text{homogeneity } (h1) \\ b\left(\vec{t} + \vec{u}, \vec{v}\right) = b\left(\vec{t}, \vec{v}\right) + b\left(\vec{u}, \vec{v}\right) & \text{additivity } (a1) \\ b\left(\vec{u}, s\vec{v}\right) = s\, b\left(\vec{u}, \vec{v}\right) & \text{homogeneity } (h2) \\ b\left(\vec{u}, \vec{v} + \vec{w}\right) = b\left(\vec{u}, \vec{v}\right) + b\left(\vec{u}, \vec{w}\right) & \text{additivity } (a2) \end{cases}$$

$$\Leftrightarrow \begin{cases} b \in \mathbb{F}^{\mathcal{V}^2} & \Leftrightarrow b : \mathcal{V} \times \mathcal{V} \to \mathbb{F} \\ b\left(\vec{u}, s\vec{v}\right) = s\, b\left(\vec{u}, \vec{v}\right) = b\left(s\vec{u}, \vec{v}\right) & \text{homogeneity } (h) \\ b\left(\vec{u}, \vec{v} + \vec{w}\right) = b\left(\vec{u}, \vec{v}\right) + b\left(\vec{u}, \vec{w}\right) & \text{additivity } (a1) \\ b\left(\vec{t} + \vec{u}, \vec{v}\right) = b\left(\vec{t}, \vec{v}\right) + b\left(\vec{u}, \vec{v}\right) & \text{additivity } (a2) \end{cases}$$

$(0,1)$-tensor
covariant
basis vector

$$\begin{cases} \vec{\widetilde{e}}_j = \vec{e}_i \widetilde{e}^i_j \\ \vec{e}_j = \vec{\widetilde{e}}_i e^i_j \end{cases}$$

covector component scalar

$$\begin{cases} \widetilde{\alpha}_j = \alpha_i \widetilde{e}^i_j \\ \alpha_j = \widetilde{\alpha}_i e^i_j \end{cases}$$

$(1,0)$-tensor
contravariant
vector component scalar

$$\begin{cases} \widetilde{v}^i = e^i_j v^j \\ v^i = \widetilde{e}^i_j \widetilde{v}^j \end{cases}$$

dual basis covector

$$\begin{cases} \overleftarrow{\widetilde{\epsilon}}^i = e^i_j \overleftarrow{\epsilon}^j \\ \overleftarrow{\epsilon}^i = \widetilde{e}^i_j \overleftarrow{\widetilde{\epsilon}}^j \end{cases}$$

$(1,1)$-tensor

linear map

$$\begin{cases} \widetilde{L}^i_j = e^i_h L^h_k \widetilde{e}^k_j \\ L^i_j = \widetilde{e}^i_h \widetilde{L}^h_k e^k_j \end{cases}$$

$(0,2)$-tensor
bilinear form $=$ 2-form

$$\begin{cases} \widetilde{b}_{ij} = b_{hk} \widetilde{e}^h_i \widetilde{e}^k_j \\ b_{ij} = \widetilde{b}_{hk} e^h_i e^k_j \end{cases}$$

sesquilinear form
tensor product

$$s \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \otimes \begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix} = s \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix} = \begin{bmatrix} s\,v^1 \\ s\,v^2 \end{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix} = \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \begin{bmatrix} s\alpha_1 & s\alpha_2 \end{bmatrix} \qquad s\left(\vec{v}\overleftarrow{\alpha}\right) = \left(s\vec{v}\right)\overleftarrow{\alpha} = \vec{v}\left(s\overleftarrow{\alpha}\right)$$

$$= \begin{bmatrix} s\,v^1 \\ s\,v^2 \end{bmatrix} \otimes \begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix} = \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \otimes \begin{bmatrix} s\alpha_1 & s\alpha_2 \end{bmatrix} \qquad s\left(\vec{v} \otimes \overleftarrow{\alpha}\right) = \left(s\vec{v}\right) \otimes \overleftarrow{\alpha} = \vec{v} \otimes \left(s\overleftarrow{\alpha}\right)$$

$$\left(\begin{bmatrix} u^1 \\ u^2 \end{bmatrix} + \begin{bmatrix} v^1 \\ v^2 \end{bmatrix}\right) \begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix} = \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix} + \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix} \qquad \left(\vec{u} + \vec{v}\right)\overleftarrow{\alpha} = \vec{u}\overleftarrow{\alpha} + \vec{v}\overleftarrow{\alpha}$$

$$\left(\begin{bmatrix} u^1 \\ u^2 \end{bmatrix} + \begin{bmatrix} v^1 \\ v^2 \end{bmatrix}\right) \otimes \begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix} = \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} \otimes \begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix} + \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \otimes \begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix} \qquad \left(\vec{u} + \vec{v}\right) \otimes \overleftarrow{\alpha} = \vec{u} \otimes \overleftarrow{\alpha} + \vec{v} \otimes \overleftarrow{\alpha}$$

$$\begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \left(\begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix} + \begin{bmatrix} \beta_1 & \beta_2 \end{bmatrix}\right) = \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix} + \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \begin{bmatrix} \beta_1 & \beta_2 \end{bmatrix} \qquad \vec{v}\left(\overleftarrow{\alpha} + \overleftarrow{\beta}\right) = \vec{v}\overleftarrow{\alpha} + \vec{v}\overleftarrow{\beta}$$

$$\begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \otimes \left(\begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix} + \begin{bmatrix} \beta_1 & \beta_2 \end{bmatrix}\right) = \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \otimes \begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix} + \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \otimes \begin{bmatrix} \beta_1 & \beta_2 \end{bmatrix} \qquad \vec{v} \otimes \left(\overleftarrow{\alpha} + \overleftarrow{\beta}\right) = \vec{v} \otimes \overleftarrow{\alpha} + \vec{v} \otimes \overleftarrow{\beta}$$

$$\begin{cases} s\left(\vec{v}\overleftarrow{\alpha}\right) = \left(s\vec{v}\right)\overleftarrow{\alpha} = \vec{v}\left(s\overleftarrow{\alpha}\right) \\ \left(\vec{u} + \vec{v}\right)\overleftarrow{\alpha} = \vec{u}\overleftarrow{\alpha} + \vec{v}\overleftarrow{\alpha} \\ \vec{v}\left(\overleftarrow{\alpha} + \overleftarrow{\beta}\right) = \vec{v}\overleftarrow{\alpha} + \vec{v}\overleftarrow{\beta} \end{cases} \Leftrightarrow \begin{cases} s\left(\vec{v} \otimes \overleftarrow{\alpha}\right) = \left(s\vec{v}\right) \otimes \overleftarrow{\alpha} = \vec{v} \otimes \left(s\overleftarrow{\alpha}\right) \\ \left(\vec{u} + \vec{v}\right) \otimes \overleftarrow{\alpha} = \vec{u} \otimes \overleftarrow{\alpha} + \vec{v} \otimes \overleftarrow{\alpha} \\ \vec{v} \otimes \left(\overleftarrow{\alpha} + \overleftarrow{\beta}\right) = \vec{v} \otimes \overleftarrow{\alpha} + \vec{v} \otimes \overleftarrow{\beta} \end{cases}$$

otimes symbol $\otimes$

$$\begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \otimes \begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix} = \begin{cases} \begin{bmatrix} v^1 \begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix} \\ v^2 \begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} v^1\alpha_1 & v^1\alpha_2 \end{bmatrix} \\ \begin{bmatrix} v^2\alpha_1 & v^2\alpha_2 \end{bmatrix} \end{bmatrix} & \text{left distribution} \\ \begin{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix}\alpha_1 & \begin{bmatrix} v^1 \\ v^2 \end{bmatrix}\alpha_2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} v^1\alpha_1 \\ v^2\alpha_1 \end{bmatrix} & \begin{bmatrix} v^1\alpha_2 \\ v^2\alpha_2 \end{bmatrix} \end{bmatrix} & \text{right distribution} \end{cases} \qquad \text{Kronecker product } (1)$$

$$\vec{v} \otimes \overleftarrow{\alpha} = \vec{e}_i v^i \otimes \alpha_j \overleftarrow{\epsilon}^j \qquad \text{tensor product with index notation } (2)$$

$$\vec{v} \otimes \overleftarrow{\alpha} \in \mathcal{V} \otimes \mathcal{V}^* \qquad \text{tensor product of vector spaces} (3)$$

otimes symbol $\otimes$ as tensor space product vs. Cartesian product or set product

$$\vec{\vec{v}} \in \mathcal{V}_1 \otimes \mathcal{V}_2 \Rightarrow \vec{\vec{v}} = \left(\vec{v}_1, \vec{v}_2\right) \text{ has bilinearity} \Leftrightarrow \begin{cases} \left(s_u\vec{u}_1 + s_v\vec{v}_1, \vec{v}_2\right) = s_u\left(\vec{u}_1, \vec{v}_2\right) + s_v\left(\vec{v}_1, \vec{v}_2\right) & (l1) \\ \left(\vec{v}_1, s_u\vec{u}_2 + s_v\vec{v}_2\right) = s_u\left(\vec{v}_1, \vec{u}_2\right) + s_v\left(\vec{v}_1, \vec{v}_2\right) & (l2) \end{cases}$$

$$e \in \mathcal{V}_1 \times \mathcal{V}_2 \Rightarrow e = \left(\vec{v}_1, \vec{v}_2\right) \text{ does not necessarily have bilinearity}$$

set via tensor space product much more than only basis tensor product as basis

$$\begin{array}{rlll}
\vec{e}_i L^i_j \overleftarrow{\epsilon}^j = L^i_j \vec{e}_i \overleftarrow{\epsilon}^j & = L^i_j \vec{e}_i \otimes \overleftarrow{\epsilon}^j & \in \mathcal{V} \otimes \mathcal{V}^* & \\
L^i_j v^j & = w^i & \mathcal{V} \to \mathcal{V} & \mathcal{V}^{\mathcal{V}} \\
\alpha_i L^i_j & = \omega_j & \mathcal{V}^* \to \mathcal{V}^* & \mathcal{V}^{*\mathcal{V}^*} \\
v^j L^i_j \alpha_i = L_j{}^i \alpha_i v^j = L^i_j \alpha_i v^j & = s & \mathcal{V}^* \times \mathcal{V} \to \mathbb{F} & \mathbb{F}^{\mathcal{V}^* \times \mathcal{V}} \\
\alpha_i L^i_j v^j = L^i{}_j v^j \alpha_i = L^i_j v^j \alpha_i & = s & \mathcal{V} \times \mathcal{V}^* \to \mathbb{F} & \mathbb{F}^{\mathcal{V} \times \mathcal{V}^*}
\end{array}$$

$$L^i_j = \begin{cases} L_j{}^i & \leftrightarrow L_j{}^i \overleftarrow{\epsilon}^j \otimes \vec{e}_i \in \mathcal{V}^* \otimes \mathcal{V} \\ L^i{}_j & \leftrightarrow L^i{}_j \vec{e}_i \otimes \overleftarrow{\epsilon}^j \in \mathcal{V} \otimes \mathcal{V}^* \end{cases}$$

$$\mathcal{V} \otimes \mathcal{V}^* \supseteq \mathcal{V}^{\mathcal{V}}, \mathcal{V}^{*\mathcal{V}^*}, \mathbb{F}^{\mathcal{V}^* \times \mathcal{V}}, \mathbb{F}^{\mathcal{V} \times \mathcal{V}^*}$$

$$\begin{array}{rlll}
b_{ij} \overleftarrow{\epsilon}^i \overleftarrow{\epsilon}^j & = b_{ij} \overleftarrow{\epsilon}^i \otimes \overleftarrow{\epsilon}^j & \in \mathcal{V}^* \otimes \mathcal{V}^* & \\
b_{ij} u^i v^j & = s & \mathcal{V} \times \mathcal{V} \to \mathbb{F} & \mathbb{F}^{\mathcal{V} \times \mathcal{V}} \\
b_{ij} v^j & = \alpha_i & \mathcal{V} \to \mathcal{V}^* & \mathcal{V}^{*\mathcal{V}} \\
b_{ij} u^i & = \beta_j & \mathcal{V} \to \mathcal{V}^* & \mathcal{V}^{*\mathcal{V}}
\end{array}$$

$$\mathcal{V} \otimes \mathcal{V}^* \supseteq \mathbb{F}^{\mathcal{V} \times \mathcal{V}}, \mathcal{V}^{*\mathcal{V}}$$

order of upper index(ices) and lower index(ices)

$$L^i_j = \begin{cases} L_j{}^i & \leftrightarrow L_j{}^i \overleftarrow{\epsilon}^j \otimes \vec{e}_i \in \mathcal{V}^* \otimes \mathcal{V} \\ L^i{}_j & \leftrightarrow L^i{}_j \vec{e}_i \otimes \overleftarrow{\epsilon}^j \in \mathcal{V} \otimes \mathcal{V}^* \end{cases}$$

$$L^i{}_j \neq L_j{}^i$$

pure (1,1)-tensor $L^i_j = v^i \alpha_j$

$$\begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \otimes \begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix} = \begin{cases} \begin{bmatrix} v^1 \begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix} \\ v^2 \begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} v^1\alpha_1 & v^1\alpha_2 \end{bmatrix} \\ \begin{bmatrix} v^2\alpha_1 & v^2\alpha_2 \end{bmatrix} \end{bmatrix} & \text{left distribution} \\ \begin{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix}\alpha_1 & \begin{bmatrix} v^1 \\ v^2 \end{bmatrix}\alpha_2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} v^1\alpha_1 \\ v^2\alpha_1 \end{bmatrix} & \begin{bmatrix} v^1\alpha_2 \\ v^2\alpha_2 \end{bmatrix} \end{bmatrix} & \text{right distribution} \end{cases}$$

$$\begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix} \otimes \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \begin{cases} \begin{bmatrix} \alpha_1\begin{bmatrix} v^1 \\ v^2 \end{bmatrix} & \alpha_2\begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \alpha_1 v^1 \\ \alpha_1 v^2 \end{bmatrix} & \begin{bmatrix} \alpha_2 v^1 \\ \alpha_2 v^2 \end{bmatrix} \end{bmatrix} & \text{left distribution} \\ \begin{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix} v^1 \\ \begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix} v^2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \alpha_1 v^1 & \alpha_2 v^1 \end{bmatrix} \\ \begin{bmatrix} \alpha_1 v^2 & \alpha_2 v^2 \end{bmatrix} \end{bmatrix} & \text{right distribution} \end{cases}$$

$$\begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \otimes \begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix} \begin{bmatrix} \overleftarrow{\epsilon}^1 \\ \overleftarrow{\epsilon}^2 \end{bmatrix} = \begin{cases} \begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix} \begin{bmatrix} v^1 \begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix} \\ v^2 \begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix} \end{bmatrix} \begin{bmatrix} \overleftarrow{\epsilon}^1 \\ \overleftarrow{\epsilon}^2 \end{bmatrix} = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} v^1\alpha_1 & v^1\alpha_2 \end{bmatrix} \\ \begin{bmatrix} v^2\alpha_1 & v^2\alpha_2 \end{bmatrix} \end{bmatrix} \begin{bmatrix} \overleftarrow{\epsilon}^1 \\ \overleftarrow{\epsilon}^2 \end{bmatrix} & \text{left distribution} \\ \begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix}\alpha_1 & \begin{bmatrix} v^1 \\ v^2 \end{bmatrix}\alpha_2 \end{bmatrix} \begin{bmatrix} \overleftarrow{\epsilon}^1 \\ \overleftarrow{\epsilon}^2 \end{bmatrix} = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} v^1\alpha_1 \\ v^2\alpha_1 \end{bmatrix} & \begin{bmatrix} v^1\alpha_2 \\ v^2\alpha_2 \end{bmatrix} \end{bmatrix} \begin{bmatrix} \overleftarrow{\epsilon}^1 \\ \overleftarrow{\epsilon}^2 \end{bmatrix} & \text{right distribution} \end{cases}$$

$$= \begin{cases} \begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} L^1{}_1 & L^1{}_2 \end{bmatrix} \\ \begin{bmatrix} L^2{}_1 & L^2{}_2 \end{bmatrix} \end{bmatrix} \begin{bmatrix} \overleftarrow{\epsilon}^1 \\ \overleftarrow{\epsilon}^2 \end{bmatrix} & \text{left distribution} \\ \begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} L^1{}_1 \\ L^2{}_1 \end{bmatrix} & \begin{bmatrix} L^1{}_2 \\ L^2{}_2 \end{bmatrix} \end{bmatrix} \begin{bmatrix} \overleftarrow{\epsilon}^1 \\ \overleftarrow{\epsilon}^2 \end{bmatrix} & \text{right distribution} \end{cases} \qquad L^i{}_j = v^i \alpha_j \leftrightarrow \vec{e}_i L^i{}_j \overleftarrow{\epsilon}^j = \vec{e}_i v^i \alpha_j \overleftarrow{\epsilon}^j$$

$$[\alpha_1 \quad \alpha_2]\begin{bmatrix}\overleftarrow{\epsilon}^1\\ \overleftarrow{\epsilon}^2\end{bmatrix}\otimes\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\begin{bmatrix}v^1\\ v^2\end{bmatrix} \overset{\text{should be}}{=} \begin{cases}\begin{bmatrix}\overleftarrow{\epsilon}^1\\ \overleftarrow{\epsilon}^2\end{bmatrix}\begin{bmatrix}\alpha_1\begin{bmatrix}v^1\\ v^2\end{bmatrix} & \alpha_2\begin{bmatrix}v^1\\ v^2\end{bmatrix}\end{bmatrix}\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}=\begin{bmatrix}\overleftarrow{\epsilon}^1\\ \overleftarrow{\epsilon}^2\end{bmatrix}\begin{bmatrix}\begin{bmatrix}\alpha_1 v^1\\ \alpha_1 v^2\end{bmatrix} & \begin{bmatrix}\alpha_2 v^1\\ \alpha_2 v^2\end{bmatrix}\end{bmatrix}\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix} & \text{left distribution}\\ \begin{bmatrix}\overleftarrow{\epsilon}^1\\ \overleftarrow{\epsilon}^2\end{bmatrix}\begin{bmatrix}\begin{bmatrix}\alpha_1 & \alpha_2\end{bmatrix}v^1\\ \begin{bmatrix}\alpha_1 & \alpha_2\end{bmatrix}v^2\end{bmatrix}\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}=\begin{bmatrix}\overleftarrow{\epsilon}^1\\ \overleftarrow{\epsilon}^2\end{bmatrix}\begin{bmatrix}\begin{bmatrix}\alpha_1 v^1 & \alpha_2 v^1\end{bmatrix}\\ \begin{bmatrix}\alpha_1 v^2 & \alpha_2 v^2\end{bmatrix}\end{bmatrix}\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix} & \text{right distribution}\end{cases}$$

$$= \begin{cases}\begin{bmatrix}\overleftarrow{\epsilon}^1\\ \overleftarrow{\epsilon}^2\end{bmatrix}\begin{bmatrix}\begin{bmatrix}L_1{}^1\\ L_1{}^2\end{bmatrix} & \begin{bmatrix}L_2{}^1\\ L_2{}^2\end{bmatrix}\end{bmatrix}\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix} & \text{left distribution}\\ \begin{bmatrix}\overleftarrow{\epsilon}^1\\ \overleftarrow{\epsilon}^2\end{bmatrix}\begin{bmatrix}\begin{bmatrix}L_1{}^1 & L_2{}^1\end{bmatrix}\\ \begin{bmatrix}L_1{}^2 & L_2{}^2\end{bmatrix}\end{bmatrix}\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix} & \text{right distribution}\end{cases} \qquad L_j{}^i = \alpha_j v^i \leftrightarrow \overleftarrow{\epsilon}^j L_j{}^i \vec{e}_i = \overleftarrow{\epsilon}^j \alpha_j v^i \vec{e}_i$$

impure (1,1)-tensor $L^i_j \not\equiv v^i \alpha_j$

$$s\left(\vec{v}\otimes\overleftarrow{\alpha}\right) = \left(s\vec{v}\right)\otimes\overleftarrow{\alpha} = \vec{v}\otimes\left(s\overleftarrow{\alpha}\right)$$

$$s\,(T\otimes U) = (s\,T)\otimes U = T\otimes(s\,U)$$

$$\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\otimes\begin{bmatrix}\overleftarrow{\epsilon}^1\\ \overleftarrow{\epsilon}^2\end{bmatrix} = \begin{cases}\begin{cases}\begin{bmatrix}\vec{e}_1\begin{bmatrix}\overleftarrow{\epsilon}^1\\ \overleftarrow{\epsilon}^2\end{bmatrix} & \vec{e}_2\begin{bmatrix}\overleftarrow{\epsilon}^1\\ \overleftarrow{\epsilon}^2\end{bmatrix}\end{bmatrix} = \begin{bmatrix}\begin{bmatrix}\vec{e}_1\overleftarrow{\epsilon}^1\\ \vec{e}_1\overleftarrow{\epsilon}^2\end{bmatrix} & \begin{bmatrix}\vec{e}_2\overleftarrow{\epsilon}^1\\ \vec{e}_2\overleftarrow{\epsilon}^2\end{bmatrix}\end{bmatrix} & \text{left distribution}\\ \begin{bmatrix}\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\overleftarrow{\epsilon}^1\\ \begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\overleftarrow{\epsilon}^2\end{bmatrix} = \begin{bmatrix}\begin{bmatrix}\vec{e}_1\overleftarrow{\epsilon}^1 & \vec{e}_2\overleftarrow{\epsilon}^1\end{bmatrix}\\ \begin{bmatrix}\vec{e}_1\overleftarrow{\epsilon}^2 & \vec{e}_2\overleftarrow{\epsilon}^2\end{bmatrix}\end{bmatrix} & \text{right distribution}\end{cases} & \otimes\text{ not preserved}\\ \begin{cases}\begin{bmatrix}\vec{e}_1\otimes\begin{bmatrix}\overleftarrow{\epsilon}^1\\ \overleftarrow{\epsilon}^2\end{bmatrix} & \vec{e}_2\otimes\begin{bmatrix}\overleftarrow{\epsilon}^1\\ \overleftarrow{\epsilon}^2\end{bmatrix}\end{bmatrix} = \begin{bmatrix}\begin{bmatrix}\vec{e}_1\otimes\overleftarrow{\epsilon}^1\\ \vec{e}_1\otimes\overleftarrow{\epsilon}^2\end{bmatrix} & \begin{bmatrix}\vec{e}_2\otimes\overleftarrow{\epsilon}^1\\ \vec{e}_2\otimes\overleftarrow{\epsilon}^2\end{bmatrix}\end{bmatrix} & \text{left distribution}\\ \begin{bmatrix}\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\otimes\overleftarrow{\epsilon}^1\\ \begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\otimes\overleftarrow{\epsilon}^2\end{bmatrix} = \begin{bmatrix}\begin{bmatrix}\vec{e}_1\otimes\overleftarrow{\epsilon}^1 & \vec{e}_2\otimes\overleftarrow{\epsilon}^1\end{bmatrix}\\ \begin{bmatrix}\vec{e}_1\otimes\overleftarrow{\epsilon}^2 & \vec{e}_2\otimes\overleftarrow{\epsilon}^2\end{bmatrix}\end{bmatrix} & \text{right distribution}\end{cases} & \otimes\text{ preserved}\end{cases}$$

$$L^i{}_j\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\otimes\begin{bmatrix}\overleftarrow{\epsilon}^1\\ \overleftarrow{\epsilon}^2\end{bmatrix} = \begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}L^i{}_j\otimes\begin{bmatrix}\overleftarrow{\epsilon}^1\\ \overleftarrow{\epsilon}^2\end{bmatrix} = \begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\otimes L^i{}_j\begin{bmatrix}\overleftarrow{\epsilon}^1\\ \overleftarrow{\epsilon}^2\end{bmatrix} = \begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\begin{bmatrix}L^1{}_1 & L^1{}_2\\ L^2{}_1 & L^2{}_2\end{bmatrix}\begin{bmatrix}\overleftarrow{\epsilon}^1\\ \overleftarrow{\epsilon}^2\end{bmatrix}$$

$$L^i{}_j\vec{e}_i\overleftarrow{\epsilon}^j = \vec{e}_i L^i{}_j\overleftarrow{\epsilon}^j = \vec{e}_i L^i{}_j\overleftarrow{\epsilon}^j$$
$$L^i{}_j\vec{e}_i\otimes\overleftarrow{\epsilon}^j = \vec{e}_i L^i{}_j\otimes\overleftarrow{\epsilon}^j = \vec{e}_i\otimes L^i{}_j\overleftarrow{\epsilon}^j$$

$$\begin{bmatrix}\overleftarrow{\epsilon}^1\\ \overleftarrow{\epsilon}^2\end{bmatrix}\otimes\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix} = \begin{cases}\begin{cases}\begin{bmatrix}\overleftarrow{\epsilon}^1\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\\ \overleftarrow{\epsilon}^2\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\end{bmatrix} = \begin{bmatrix}\begin{bmatrix}\overleftarrow{\epsilon}^1\vec{e}_1 & \overleftarrow{\epsilon}^1\vec{e}_2\end{bmatrix}\\ \begin{bmatrix}\overleftarrow{\epsilon}^2\vec{e}_1 & \overleftarrow{\epsilon}^2\vec{e}_2\end{bmatrix}\end{bmatrix} = \begin{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix}\\ \begin{bmatrix}0 & 1\end{bmatrix}\end{bmatrix} & \text{left distribution}\\ \begin{bmatrix}\begin{bmatrix}\overleftarrow{\epsilon}^1\\ \overleftarrow{\epsilon}^2\end{bmatrix}\vec{e}_1 & \begin{bmatrix}\overleftarrow{\epsilon}^1\\ \overleftarrow{\epsilon}^2\end{bmatrix}\vec{e}_2\end{bmatrix} = \begin{bmatrix}\begin{bmatrix}\overleftarrow{\epsilon}^1\vec{e}_1\\ \overleftarrow{\epsilon}^2\vec{e}_1\end{bmatrix} & \begin{bmatrix}\overleftarrow{\epsilon}^1\vec{e}_2\\ \overleftarrow{\epsilon}^2\vec{e}_2\end{bmatrix}\end{bmatrix} = \begin{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} & \begin{bmatrix}0\\ 1\end{bmatrix}\end{bmatrix} & \text{right distribution}\end{cases} & \otimes\text{ not preserved}\\ \begin{cases}\begin{bmatrix}\overleftarrow{\epsilon}^1\otimes\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\\ \overleftarrow{\epsilon}^2\otimes\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\end{bmatrix} = \begin{bmatrix}\begin{bmatrix}\overleftarrow{\epsilon}^1\otimes\vec{e}_1 & \overleftarrow{\epsilon}^1\otimes\vec{e}_2\end{bmatrix}\\ \begin{bmatrix}\overleftarrow{\epsilon}^2\otimes\vec{e}_1 & \overleftarrow{\epsilon}^2\otimes\vec{e}_2\end{bmatrix}\end{bmatrix} & \text{left distribution}\\ \begin{bmatrix}\begin{bmatrix}\overleftarrow{\epsilon}^1\\ \overleftarrow{\epsilon}^2\end{bmatrix}\otimes\vec{e}_1 & \begin{bmatrix}\overleftarrow{\epsilon}^1\\ \overleftarrow{\epsilon}^2\end{bmatrix}\otimes\vec{e}_2\end{bmatrix} = \begin{bmatrix}\begin{bmatrix}\overleftarrow{\epsilon}^1\otimes\vec{e}_1\\ \overleftarrow{\epsilon}^2\otimes\vec{e}_1\end{bmatrix} & \begin{bmatrix}\overleftarrow{\epsilon}^1\otimes\vec{e}_2\\ \overleftarrow{\epsilon}^2\otimes\vec{e}_2\end{bmatrix}\end{bmatrix} & \text{right distribution}\end{cases} & \otimes\text{ preserved}\end{cases}$$

$$L_j{}^i\begin{bmatrix}\overleftarrow{\epsilon}^1\\ \overleftarrow{\epsilon}^2\end{bmatrix}\otimes\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix} = \begin{bmatrix}\overleftarrow{\epsilon}^1\\ \overleftarrow{\epsilon}^2\end{bmatrix}L_j{}^i\otimes\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix} = \begin{bmatrix}\overleftarrow{\epsilon}^1\\ \overleftarrow{\epsilon}^2\end{bmatrix}\otimes L_j{}^i\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix} = \begin{bmatrix}\overleftarrow{\epsilon}^1\\ \overleftarrow{\epsilon}^2\end{bmatrix}\begin{bmatrix}L_1{}^1 & L_2{}^1\\ L_1{}^2 & L_2{}^2\end{bmatrix}\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}$$

$$L_j{}^i\overleftarrow{\epsilon}^j\vec{e}_i = \overleftarrow{\epsilon}^j L_j{}^i\vec{e}_i = \overleftarrow{\epsilon}^j L_j{}^i\vec{e}_i$$
$$L_j{}^i\overleftarrow{\epsilon}^j\otimes\vec{e}_i = \overleftarrow{\epsilon}^j L_j{}^i\otimes\vec{e}_i = \overleftarrow{\epsilon}^j\otimes L_j{}^i\vec{e}_i$$

forward and backward transformation actually a (1,1)-tensor $\in \mathcal{V}\otimes\mathcal{V}^*$

$$F = \left[\widetilde{E}\right]_E = \left[\widetilde{e}\right]_{\vec{e}} = \left[\widetilde{e}^i_j\right] \overset{\text{e.g.}}{=} \begin{bmatrix}\widetilde{e}^1_1 & \widetilde{e}^1_2\\ \widetilde{e}^2_1 & \widetilde{e}^2_2\end{bmatrix}$$
$$= \left[\widetilde{e}^i{}_j\right] \overset{\text{e.g.}}{=} \begin{bmatrix}\widetilde{e}^1{}_1 & \widetilde{e}^1{}_2\\ \widetilde{e}^2{}_1 & \widetilde{e}^2{}_2\end{bmatrix}$$
$$\updownarrow$$
$$E\left[\widetilde{E}\right]_E^{\mathcal{E}}\mathcal{E} = \left[\vec{e}\right]\left[\widetilde{e}\right]_{\vec{e}}^{\overleftarrow{\epsilon}}\left[\overleftarrow{\epsilon}\right] = \left[\vec{e}_i\right]\left[\widetilde{e}^i{}_j\right]\left[\overleftarrow{\epsilon}^j\right] \overset{\text{e.g.}}{=} \begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\begin{bmatrix}\widetilde{e}^1{}_1 & \widetilde{e}^1{}_2\\ \widetilde{e}^2{}_1 & \widetilde{e}^2{}_2\end{bmatrix}\begin{bmatrix}\overleftarrow{\epsilon}^1\\ \overleftarrow{\epsilon}^2\end{bmatrix}$$

$$B = [E]_{\widetilde{E}} = [e]_{\widetilde{\vec{e}}} = \left[e^i_j\right] \overset{\text{e.g.}}{=} \begin{bmatrix} e^1_1 & e^1_2 \\ e^2_1 & e^2_2 \end{bmatrix}$$

$$= \left[e^i{}_j\right] \overset{\text{e.g.}}{=} \begin{bmatrix} e^1{}_1 & e^1{}_2 \\ e^2{}_1 & e^2{}_2 \end{bmatrix}$$

$$\updownarrow$$

$$\widetilde{E}\,[E]^{\widetilde{\mathcal{E}}}_{\widetilde{E}}\,\widetilde{\mathcal{E}} = \left[\widetilde{\vec{e}}\right]\,[e]^{\widetilde{\overleftarrow{\epsilon}}}_{\widetilde{\vec{e}}}\,\left[\widetilde{\overleftarrow{\epsilon}}\right] = \left[\widetilde{\vec{e}}_i\right]\,\left[e^i{}_j\right]\,\left[\widetilde{\overleftarrow{\epsilon}}^j\right] \overset{\text{e.g.}}{=} \begin{bmatrix} \widetilde{\vec{e}}_1 & \widetilde{\vec{e}}_2 \end{bmatrix} \begin{bmatrix} e^1{}_1 & e^1{}_2 \\ e^2{}_1 & e^2{}_2 \end{bmatrix} \begin{bmatrix} \widetilde{\overleftarrow{\epsilon}}^1 \\ \widetilde{\overleftarrow{\epsilon}}^2 \end{bmatrix}$$

$(0,1)$-tensor
covariant
basis vector

$$\begin{cases} \widetilde{\vec{e}}_j = \vec{e}_i \widetilde{e}^i_j \\ \vec{e}_j = \widetilde{\vec{e}}_i e^i_j \end{cases}$$

covector component scalar

$$\begin{cases} \widetilde{\alpha}_j = \alpha_i \widetilde{e}^i_j \\ \alpha_j = \widetilde{\alpha}_i e^i_j \end{cases}$$

$(1,0)$-tensor
contravariant
vector component scalar

$$\begin{cases} \widetilde{v}^i = e^i_j v^j \\ v^i = \widetilde{e}^i_j \widetilde{v}^j \end{cases}$$

dual basis covector

$$\begin{cases} \widetilde{\overleftarrow{\epsilon}}^i = e^i_j \overleftarrow{\epsilon}^j \\ \overleftarrow{\epsilon}^i = \widetilde{e}^i_j \widetilde{\overleftarrow{\epsilon}}^j \end{cases}$$

$(1,1)$-tensor

linear map

$$\begin{cases} \widetilde{L}^i_j = e^i_h L^h_k \widetilde{e}^k_j \\ L^i_j = \widetilde{e}^i_h \widetilde{L}^h_k e^k_j \end{cases}$$

$(0,2)$-tensor
bilinear form = 2-form

$$\begin{cases} \widetilde{b}_{ij} = b_{hk} \widetilde{e}^h_i \widetilde{e}^k_j \\ b_{ij} = \widetilde{b}_{hk} e^h_i e^k_j \end{cases}$$

$$\begin{bmatrix} \widetilde{\vec{e}}_1 & \widetilde{\vec{e}}_2 \end{bmatrix} = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix} \begin{bmatrix} \widetilde{e}^1_1 & \widetilde{e}^1_2 \\ \widetilde{e}^2_1 & \widetilde{e}^2_2 \end{bmatrix} \qquad \begin{bmatrix} \widetilde{\vec{e}}_1 \cdot \\ \widetilde{\vec{e}}_2 \cdot \end{bmatrix} = \begin{bmatrix} \widetilde{e}^1_1 & \widetilde{e}^1_2 \\ \widetilde{e}^2_1 & \widetilde{e}^2_2 \end{bmatrix}^{\intercal} \begin{bmatrix} \vec{e}_1 \cdot \\ \vec{e}_2 \cdot \end{bmatrix}$$

$$\begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix} = \begin{bmatrix} \widetilde{\vec{e}}_1 & \widetilde{\vec{e}}_2 \end{bmatrix} \begin{bmatrix} e^1_1 & e^1_2 \\ e^2_1 & e^2_2 \end{bmatrix} \qquad \begin{bmatrix} \vec{e}_1 \cdot \\ \vec{e}_2 \cdot \end{bmatrix} = \begin{bmatrix} e^1_1 & e^1_2 \\ e^2_1 & e^2_2 \end{bmatrix}^{\intercal} \begin{bmatrix} \widetilde{\vec{e}}_1 \cdot \\ \widetilde{\vec{e}}_2 \cdot \end{bmatrix}$$

Table 1.1: transformation as unordered tensor

$(0,1)$-tensor
covariant
basis vector

$$\begin{cases} \widetilde{\vec{e}}_j = \vec{e}_i \widetilde{e}^i{}_j \\ \vec{e}_j = \widetilde{\vec{e}}_i e^i{}_j \end{cases}$$

covector component scalar

$$\begin{cases} \widetilde{\alpha}_j = \alpha_i \widetilde{e}^i{}_j \\ \alpha_j = \widetilde{\alpha}_i e^i{}_j \end{cases}$$

$(1,0)$-tensor
contravariant
vector component scalar

$$\begin{cases} \widetilde{v}^i = e^i{}_j v^j \\ v^i = \widetilde{e}^i{}_j \widetilde{v}^j \end{cases}$$

dual basis covector

$$\begin{cases} \widetilde{\overleftarrow{\epsilon}}^i = e^i{}_j \overleftarrow{\epsilon}^j \\ \overleftarrow{\epsilon}^i = \widetilde{e}^i{}_j \widetilde{\overleftarrow{\epsilon}}^j \end{cases}$$

$(1,1)$-tensor

linear map

$$\begin{cases} \widetilde{L}^i{}_j = e^i{}_h L^h{}_k \widetilde{e}^k{}_j \\ L^i{}_j = \widetilde{e}^i{}_h \widetilde{L}^h{}_k e^k{}_j \end{cases}$$

$(0,2)$-tensor
bilinear form = 2-form: including metric tensor

$$\begin{cases} \widetilde{b}_{ij} = b_{hk} \widetilde{e}^h{}_i \widetilde{e}^k{}_j \\ b_{ij} = \widetilde{b}_{hk} e^h{}_i e^k{}_j \end{cases}$$

$$\begin{bmatrix} \widetilde{\vec{e}}_1 & \widetilde{\vec{e}}_2 \end{bmatrix} = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix} \begin{bmatrix} \widetilde{e}^1{}_1 & \widetilde{e}^1{}_2 \\ \widetilde{e}^2{}_1 & \widetilde{e}^2{}_2 \end{bmatrix} \qquad \begin{bmatrix} \widetilde{\vec{e}}_1 \cdot \\ \widetilde{\vec{e}}_2 \cdot \end{bmatrix} = \begin{bmatrix} \widetilde{e}^1{}_1 & \widetilde{e}^1{}_2 \\ \widetilde{e}^2{}_1 & \widetilde{e}^2{}_2 \end{bmatrix}^{\intercal} \begin{bmatrix} \vec{e}_1 \cdot \\ \vec{e}_2 \cdot \end{bmatrix}$$

$$\begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix} = \begin{bmatrix} \widetilde{\vec{e}}_1 & \widetilde{\vec{e}}_2 \end{bmatrix} \begin{bmatrix} e^1{}_1 & e^1{}_2 \\ e^2{}_1 & e^2{}_2 \end{bmatrix} \qquad \begin{bmatrix} \vec{e}_1 \cdot \\ \vec{e}_2 \cdot \end{bmatrix} = \begin{bmatrix} e^1{}_1 & e^1{}_2 \\ e^2{}_1 & e^2{}_2 \end{bmatrix}^{\intercal} \begin{bmatrix} \widetilde{\vec{e}}_1 \cdot \\ \widetilde{\vec{e}}_2 \cdot \end{bmatrix}$$

Table 1.2: transformation as ordered tensor

multilinear map

$$T$$

rank-0 tensor

$$s \in \mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \cdots\} \qquad \text{0-tensor}$$

$$\binom{0}{0} = 1$$

rank-1 tensor

$$\begin{array}{lll} T^i \leftrightarrow T^i \vec{e}_i & \in \mathcal{V} & (1,0)\text{-tensor} \\ T_i \leftrightarrow T_i \overleftarrow{\epsilon}^i & \in \mathcal{V}^* & (0,1)\text{-tensor} \end{array}$$

$$\binom{1}{0} + \binom{1}{1} = 1 + 1 = 2$$

rank-2 tensor

$$\begin{array}{llr} T^{ij} \leftrightarrow T^{ij} \vec{e}_i \otimes \vec{e}_j & \in \mathcal{V} \otimes \mathcal{V} & (2,0)\text{-tensor} \\ T^i{}_j \leftrightarrow T^i{}_j \vec{e}_i \otimes \overleftarrow{\epsilon}^j & \in \mathcal{V} \otimes \mathcal{V}^* & (1,1)\text{-tensor} \\ T_j{}^i \leftrightarrow T_j{}^i \overleftarrow{\epsilon}^j \otimes \vec{e}_i & \in \mathcal{V}^* \otimes \mathcal{V} & \text{transpose } (1,1)\text{-tensor} \\ T_{ij} \leftrightarrow T_{ij} \overleftarrow{\epsilon}^i \otimes \overleftarrow{\epsilon}^j & \in \mathcal{V}^* \otimes \mathcal{V}^* & (0,2)\text{-tensor} \end{array}$$

$$\binom{2}{0} + \binom{2}{1} + \binom{2}{2} = 1 + 2 + 1 = 4$$

metric tensor

$$[[g_{11} \quad g_{12}] \quad [g_{21} \quad g_{22}]] \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \equiv \begin{bmatrix} u^1 & u^2 \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix}$$

$$[[g_{11} \quad g_{12}] \quad [g_{21} \quad g_{22}]] \begin{bmatrix} \overleftarrow{\epsilon}^1 \\ \overleftarrow{\epsilon}^2 \end{bmatrix} \begin{bmatrix} \overleftarrow{\epsilon}^1 \\ \overleftarrow{\epsilon}^2 \end{bmatrix} \begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} \begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \equiv \begin{bmatrix} u^1 & u^2 \end{bmatrix} \begin{bmatrix} \vec{e}_1 \cdot \vec{e}_1 & \vec{e}_1 \cdot \vec{e}_2 \\ \vec{e}_2 \cdot \vec{e}_1 & \vec{e}_2 \cdot \vec{e}_2 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix}$$

$$\begin{aligned} &\begin{bmatrix} u^1 & u^2 \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \\ &= \begin{bmatrix} u^1 & u^2 \end{bmatrix} \begin{bmatrix} \vec{e}_1 \cdot \vec{e}_1 & \vec{e}_1 \cdot \vec{e}_2 \\ \vec{e}_2 \cdot \vec{e}_1 & \vec{e}_2 \cdot \vec{e}_2 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \\ &= \begin{bmatrix} u^1 & u^2 \end{bmatrix} \begin{bmatrix} \vec{e}_1 \cdot \\ \vec{e}_2 \cdot \end{bmatrix} \begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \end{aligned}$$

$$g_{ij} = \vec{e}_i \cdot \vec{e}_j$$

$$\begin{bmatrix} \vec{e}_1 \cdot \\ \vec{e}_2 \cdot \end{bmatrix} \begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix} = \begin{bmatrix} \vec{e}_1 \cdot \vec{e}_1 & \vec{e}_1 \cdot \vec{e}_2 \\ \vec{e}_2 \cdot \vec{e}_1 & \vec{e}_2 \cdot \vec{e}_2 \end{bmatrix}$$

$$\begin{aligned} g\left(\vec{u}, \vec{v}\right) &= [[g_{11} \quad g_{12}] \quad [g_{21} \quad g_{22}]] \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \\ = g\left(\vec{v}, \vec{u}\right) &= [[g_{11} \quad g_{12}] \quad [g_{21} \quad g_{22}]] \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} \\ &= [[g_{11} \quad g_{21}] \quad [g_{12} \quad g_{22}]] \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} \end{aligned}$$

$$g\left(\vec{u}, \vec{v}\right) = g\left(\vec{v}, \vec{u}\right) \Leftrightarrow g_{ij} = g_{ji}$$

$$g_{21} = g_{12} \Leftarrow g_{ij} = g_{ji}$$

$$= g\vec{u}\vec{v} = g_{ij} \overleftarrow{\epsilon}^i \overleftarrow{\epsilon}^j \vec{e}_k u^k \vec{e}_\ell v^\ell \qquad = \begin{cases} g_{ij} \overleftarrow{\epsilon}^i \vec{e}_k u^k \overleftarrow{\epsilon}^j \vec{e}_\ell v^\ell = g_{ij} \delta^i{}_k u^k \delta^j{}_\ell v^\ell \\ g_{ij} \overleftarrow{\epsilon}^i \vec{e}_\ell v^\ell \overleftarrow{\epsilon}^j \vec{e}_k u^k = g_{ij} \delta^i{}_\ell v^\ell \delta^j{}_k u^k \end{cases}$$

$$= g\vec{v}\vec{u} = g_{ij} \overleftarrow{\epsilon}^i \overleftarrow{\epsilon}^j \vec{e}_k v^k \vec{e}_\ell u^\ell \qquad = \begin{cases} g_{ij} \overleftarrow{\epsilon}^i \vec{e}_k v^k \overleftarrow{\epsilon}^j \vec{e}_\ell u^\ell = g_{ij} \delta^i{}_k v^k \delta^j{}_\ell u^\ell \\ g_{ij} \overleftarrow{\epsilon}^i \vec{e}_\ell u^\ell \overleftarrow{\epsilon}^j \vec{e}_k v^k = g_{ij} \delta^i{}_\ell u^\ell \delta^j{}_k v^k \end{cases}$$

compared to metric tensor

$$g = [[g_{11} \quad g_{12}] \quad [g_{21} \quad g_{22}]] \leftrightarrow [[g_{11} \quad g_{12}] \quad [g_{21} \quad g_{22}]] \begin{bmatrix} \overleftarrow{\epsilon}^1 \\ \overleftarrow{\epsilon}^2 \end{bmatrix} \begin{bmatrix} \overleftarrow{\epsilon}^1 \\ \overleftarrow{\epsilon}^2 \end{bmatrix}$$

inverse metric tensor

$$\mathfrak{g} = \begin{bmatrix} \begin{bmatrix} \mathfrak{g}^{11} \\ \mathfrak{g}^{12} \end{bmatrix} \\ \begin{bmatrix} \mathfrak{g}^{21} \\ \mathfrak{g}^{22} \end{bmatrix} \end{bmatrix} \leftrightarrow \begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix} \begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \mathfrak{g}^{11} \\ \mathfrak{g}^{12} \end{bmatrix} \\ \begin{bmatrix} \mathfrak{g}^{21} \\ \mathfrak{g}^{22} \end{bmatrix} \end{bmatrix}$$

$$
\begin{aligned}
g\mathfrak{g} &= \begin{bmatrix}\begin{bmatrix}g_{11} & g_{12}\end{bmatrix} & \begin{bmatrix}g_{21} & g_{22}\end{bmatrix}\end{bmatrix}\begin{bmatrix}\begin{bmatrix}\mathfrak{g}^{11}\\ \mathfrak{g}^{12}\end{bmatrix}\\ \begin{bmatrix}\mathfrak{g}^{21}\\ \mathfrak{g}^{22}\end{bmatrix}\end{bmatrix}\\
&= \begin{bmatrix}g_{11} & g_{12}\end{bmatrix}\begin{bmatrix}\mathfrak{g}^{11}\\ \mathfrak{g}^{12}\end{bmatrix} + \begin{bmatrix}g_{21} & g_{22}\end{bmatrix}\begin{bmatrix}\mathfrak{g}^{21}\\ \mathfrak{g}^{22}\end{bmatrix}\\
&= g_{11}\mathfrak{g}^{11} + g_{12}\mathfrak{g}^{12} + g_{21}\mathfrak{g}^{21} + g_{22}\mathfrak{g}^{22}\\
&= g_{11}\mathfrak{g}^{11} + g_{21}\mathfrak{g}^{12} + g_{12}\mathfrak{g}^{21} + g_{22}\mathfrak{g}^{22} && g_{21} = g_{12} \Leftarrow g_{ij} = g_{ji}\\
&= \mathfrak{g}^{11}g_{11} + \mathfrak{g}^{12}g_{21} + \mathfrak{g}^{21}g_{12} + \mathfrak{g}^{22}g_{22}\\
&= (\mathfrak{g}^{11}g_{11} + \mathfrak{g}^{12}g_{21}) + (\mathfrak{g}^{21}g_{12} + \mathfrak{g}^{22}g_{22})\\
&= \delta^{1}{}_{1} + \delta^{2}{}_{2} && \mathfrak{g}^{ik}g_{kj} = \delta^{i}{}_{j}\\
&= 1 + 1 = 2
\end{aligned}
$$

$$
\begin{aligned}
g\mathfrak{g} &= g_{ij}\overleftarrow{\epsilon}^{i}\overleftarrow{\epsilon}^{j}\overrightarrow{e}_{k}\overrightarrow{e}_{\ell}\mathfrak{g}^{k\ell}\\
&= g_{ij}\overleftarrow{\epsilon}^{i}\overrightarrow{e}_{k}\overleftarrow{\epsilon}^{j}\overrightarrow{e}_{\ell}\mathfrak{g}^{k\ell}\\
&= g_{ij}\delta^{i}{}_{k}\delta^{j}{}_{\ell}\mathfrak{g}^{k\ell}\\
&= g_{kj}\mathfrak{g}^{kj}
\end{aligned}
$$

$$
\begin{aligned}
&= g_{ij}\overleftarrow{\epsilon}^{i}\delta^{j}{}_{k}\overrightarrow{e}_{\ell}\mathfrak{g}^{k\ell}\\
&= g_{ik}\overleftarrow{\epsilon}^{i}\overrightarrow{e}_{\ell}\mathfrak{g}^{k\ell}\\
&= g_{ik}\delta^{i}{}_{\ell}\mathfrak{g}^{k\ell}\\
&= g_{ik}\mathfrak{g}^{ki}\\
&= \mathfrak{g}^{ki}g_{ik}\\
&= \sum_{k=1}^{2}\sum_{i=1}^{2}\mathfrak{g}^{ki}g_{ik} = 2
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{g}\,g &= \begin{bmatrix}\begin{bmatrix}\mathfrak{g}^{11}\\ \mathfrak{g}^{12}\end{bmatrix}\\ \begin{bmatrix}\mathfrak{g}^{21}\\ \mathfrak{g}^{22}\end{bmatrix}\end{bmatrix}\begin{bmatrix}\begin{bmatrix}g_{11} & g_{12}\end{bmatrix} & \begin{bmatrix}g_{21} & g_{22}\end{bmatrix}\end{bmatrix} = \begin{bmatrix}\begin{bmatrix}\mathfrak{g}^{11}\\ \mathfrak{g}^{12}\end{bmatrix}\begin{bmatrix}\begin{bmatrix}g_{11} & g_{12}\end{bmatrix} & \begin{bmatrix}g_{21} & g_{22}\end{bmatrix}\end{bmatrix}\\ \begin{bmatrix}\mathfrak{g}^{21}\\ \mathfrak{g}^{22}\end{bmatrix}\begin{bmatrix}\begin{bmatrix}g_{11} & g_{12}\end{bmatrix} & \begin{bmatrix}g_{21} & g_{22}\end{bmatrix}\end{bmatrix}\end{bmatrix}\\
&= \begin{bmatrix}\begin{bmatrix}\begin{bmatrix}\mathfrak{g}^{11}\\ \mathfrak{g}^{12}\end{bmatrix}\begin{bmatrix}g_{11} & g_{12}\end{bmatrix}\\ \begin{bmatrix}\mathfrak{g}^{21}\\ \mathfrak{g}^{22}\end{bmatrix}\begin{bmatrix}g_{11} & g_{12}\end{bmatrix}\end{bmatrix} & \begin{bmatrix}\begin{bmatrix}\mathfrak{g}^{11}\\ \mathfrak{g}^{12}\end{bmatrix}\begin{bmatrix}g_{21} & g_{22}\end{bmatrix}\\ \begin{bmatrix}\mathfrak{g}^{21}\\ \mathfrak{g}^{22}\end{bmatrix}\begin{bmatrix}g_{21} & g_{22}\end{bmatrix}\end{bmatrix}\end{bmatrix} = \begin{bmatrix}\begin{bmatrix}\begin{bmatrix}\mathfrak{g}^{11}\\ \mathfrak{g}^{12}\\ \mathfrak{g}^{21}\\ \mathfrak{g}^{22}\end{bmatrix}\begin{bmatrix}g_{11} & g_{12}\\ g_{11} & g_{12}\\ g_{11} & g_{12}\\ g_{11} & g_{12}\end{bmatrix}\end{bmatrix} & \begin{bmatrix}\begin{bmatrix}\mathfrak{g}^{11}\\ \mathfrak{g}^{12}\\ \mathfrak{g}^{21}\\ \mathfrak{g}^{22}\end{bmatrix}\begin{bmatrix}g_{21} & g_{22}\\ g_{21} & g_{22}\\ g_{21} & g_{22}\\ g_{21} & g_{22}\end{bmatrix}\end{bmatrix}\end{bmatrix}\\
&= \begin{bmatrix}\begin{bmatrix}\begin{bmatrix}\mathfrak{g}^{11}g_{11} & \mathfrak{g}^{11}g_{12}\\ \mathfrak{g}^{12}g_{11} & \mathfrak{g}^{12}g_{12}\\ \mathfrak{g}^{21}g_{11} & \mathfrak{g}^{21}g_{12}\\ \mathfrak{g}^{22}g_{11} & \mathfrak{g}^{22}g_{12}\end{bmatrix}\end{bmatrix} & \begin{bmatrix}\begin{bmatrix}\mathfrak{g}^{11}g_{21} & \mathfrak{g}^{11}g_{22}\\ \mathfrak{g}^{12}g_{21} & \mathfrak{g}^{12}g_{22}\\ \mathfrak{g}^{21}g_{21} & \mathfrak{g}^{21}g_{22}\\ \mathfrak{g}^{22}g_{21} & \mathfrak{g}^{22}g_{22}\end{bmatrix}\end{bmatrix}\end{bmatrix}
\end{aligned}
$$

$$
\begin{bmatrix}\begin{bmatrix}g_{11} & g_{12}\end{bmatrix} & \begin{bmatrix}g_{21} & g_{22}\end{bmatrix}\end{bmatrix}\begin{bmatrix}u^{1}\\ u^{2}\end{bmatrix}\begin{bmatrix}v^{1}\\ v^{2}\end{bmatrix} \equiv \begin{bmatrix}u^{1} & u^{2}\end{bmatrix}\begin{bmatrix}g_{11} & g_{12}\\ g_{21} & g_{22}\end{bmatrix}\begin{bmatrix}v^{1}\\ v^{2}\end{bmatrix}
$$

$$\begin{bmatrix}u^1 & u^2\end{bmatrix} g \begin{bmatrix}v^1\\ v^2\end{bmatrix} = \begin{bmatrix}u^1 & u^2\end{bmatrix} \begin{bmatrix}\vec{e}_1\cdot\vec{e}_1 & \vec{e}_1\cdot\vec{e}_2\\ \vec{e}_2\cdot\vec{e}_1 & \vec{e}_2\cdot\vec{e}_2\end{bmatrix}\begin{bmatrix}v^1\\ v^2\end{bmatrix} = \begin{bmatrix}u^1 & u^2\end{bmatrix}\begin{bmatrix}g_{11} & g_{12}\\ g_{21} & g_{22}\end{bmatrix}\begin{bmatrix}v^1\\ v^2\end{bmatrix}$$

$$= \begin{bmatrix}u^1 & u^2\end{bmatrix}\begin{bmatrix}\vec{e}_1\cdot\\ \vec{e}_2\cdot\end{bmatrix}\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\begin{bmatrix}v^1\\ v^2\end{bmatrix} = \begin{bmatrix}u^1 & u^2\end{bmatrix}\begin{bmatrix}\vec{e}_1\cdot\\ \vec{e}_2\cdot\end{bmatrix}\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\begin{bmatrix}v^1\\ v^2\end{bmatrix}$$

$$= u^1v^1\vec{e}_1\cdot\vec{e}_1 + u^1v^2\vec{e}_1\cdot\vec{e}_2 + u^2v^1\vec{e}_2\cdot\vec{e}_1 + u^2v^2\vec{e}_2\cdot\vec{e}_2$$

$$= u^1\vec{e}_1\cdot\vec{e}_1v^1 + u^1\vec{e}_1\cdot\vec{e}_2v^2 + u^2\vec{e}_2\cdot\vec{e}_1v^1 + u^2\vec{e}_2\cdot\vec{e}_2v^2$$

$$= \vec{e}_1\cdot\vec{e}_1u^1v^1 + \vec{e}_1\cdot\vec{e}_2u^1v^2 + \vec{e}_2\cdot\vec{e}_1u^2v^1 + \vec{e}_2\cdot\vec{e}_2u^2v^2$$

$$= u^1g_{11}v^1 + u^1g_{12}v^2 + u^2g_{21}v^1 + u^2g_{22}v^2$$

$$= g_{11}u^1v^1 + g_{12}u^1v^2 + g_{21}u^2v^1 + g_{22}u^2v^2$$

$$\equiv g\begin{bmatrix}u^1\\ u^2\end{bmatrix}\begin{bmatrix}v^1\\ v^2\end{bmatrix} = \begin{bmatrix}\begin{bmatrix}g_{11} & g_{12}\end{bmatrix} & \begin{bmatrix}g_{21} & g_{22}\end{bmatrix}\end{bmatrix}\begin{bmatrix}u^1\\ u^2\end{bmatrix}\begin{bmatrix}v^1\\ v^2\end{bmatrix} = \left(\begin{bmatrix}g_{11} & g_{12}\end{bmatrix}u^1 + \begin{bmatrix}g_{21} & g_{22}\end{bmatrix}u^2\right)\begin{bmatrix}v^1\\ v^2\end{bmatrix}$$

$$= \left(\begin{bmatrix}g_{11}u^1 & g_{12}u^1\end{bmatrix} + \begin{bmatrix}g_{21}u^2 & g_{22}u^2\end{bmatrix}\right)\begin{bmatrix}v^1\\ v^2\end{bmatrix} = \begin{bmatrix}g_{11}u^1 + g_{21}u^2 & g_{12}u^1 + g_{22}u^2\end{bmatrix}\begin{bmatrix}v^1\\ v^2\end{bmatrix}$$

$$= \left(g_{11}u^1 + g_{21}u^2\right)v^1 + \left(g_{12}u^1 + g_{22}u^2\right)v^2 = \left(g_{11}u^1v^1 + g_{21}u^2v^1\right) + \left(g_{12}u^1v^2 + g_{22}u^2v^2\right)$$

$$= g_{11}u^1v^1 + g_{21}u^2v^1 + g_{12}u^1v^2 + g_{22}u^2v^2$$

$$= \begin{bmatrix}\begin{bmatrix}g_{11} & g_{12}\end{bmatrix} & \begin{bmatrix}g_{21} & g_{22}\end{bmatrix}\end{bmatrix}\begin{bmatrix}u^1\\ u^2\end{bmatrix}\begin{bmatrix}v^1\\ v^2\end{bmatrix} = \begin{bmatrix}\begin{bmatrix}\vec{e}_1\cdot\vec{e}_1 & \vec{e}_1\cdot\vec{e}_2\end{bmatrix} & \begin{bmatrix}\vec{e}_2\cdot\vec{e}_1 & \vec{e}_2\cdot\vec{e}_2\end{bmatrix}\end{bmatrix}\begin{bmatrix}u^1\\ u^2\end{bmatrix}\begin{bmatrix}v^1\\ v^2\end{bmatrix}$$

$$= \begin{bmatrix}\begin{bmatrix}\vec{e}_1\cdot\vec{e}_1 & \vec{e}_1\cdot\vec{e}_2\end{bmatrix} & \begin{bmatrix}\vec{e}_2\cdot\vec{e}_1 & \vec{e}_2\cdot\vec{e}_2\end{bmatrix}\end{bmatrix}\begin{bmatrix}u^1\\ u^2\end{bmatrix}\begin{bmatrix}v^1\\ v^2\end{bmatrix} = \begin{bmatrix}\vec{e}_1\cdot\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix} & \vec{e}_2\cdot\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\end{bmatrix}\begin{bmatrix}u^1\\ u^2\end{bmatrix}\begin{bmatrix}v^1\\ v^2\end{bmatrix}$$

$$= \begin{bmatrix}\vec{e}_1\cdot & \vec{e}_2\cdot\end{bmatrix}\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\begin{bmatrix}u^1\\ u^2\end{bmatrix}\begin{bmatrix}v^1\\ v^2\end{bmatrix}$$

$$\begin{bmatrix}\begin{bmatrix}g_{11} & g_{12}\end{bmatrix} & \begin{bmatrix}g_{21} & g_{22}\end{bmatrix}\end{bmatrix}\begin{bmatrix}u^1\\ u^2\end{bmatrix}\begin{bmatrix}v^1\\ v^2\end{bmatrix} \equiv \begin{bmatrix}u^1 & u^2\end{bmatrix}\begin{bmatrix}g_{11} & g_{12}\\ g_{21} & g_{22}\end{bmatrix}\begin{bmatrix}v^1\\ v^2\end{bmatrix}$$

$$\begin{bmatrix}\begin{bmatrix}g_{11} & g_{12}\end{bmatrix} & \begin{bmatrix}g_{21} & g_{22}\end{bmatrix}\end{bmatrix}\square\begin{bmatrix}v^1\\ v^2\end{bmatrix} \stackrel{?}{=} \square\begin{bmatrix}g_{11} & g_{12}\\ g_{21} & g_{22}\end{bmatrix}\begin{bmatrix}v^1\\ v^2\end{bmatrix}$$

$$g_{11}\square^1v^1 + g_{21}\square^2v^1 + g_{12}\square^1v^2 + g_{22}\square^2v^2 \quad \square\begin{bmatrix}\begin{bmatrix}g_{11} & g_{12}\end{bmatrix}\begin{bmatrix}v^1\\ v^2\end{bmatrix}\\ \begin{bmatrix}g_{21} & g_{22}\end{bmatrix}\begin{bmatrix}v^1\\ v^2\end{bmatrix}\end{bmatrix}$$

$$g_{11}\square^1v^1 + g_{21}\square^2v^1 + g_{12}\square^1v^2 + g_{22}\square^2v^2 \quad \square\begin{bmatrix}g_{11}v^1 + g_{12}v^2\\ g_{21}v^1 + g_{22}v^2\end{bmatrix}$$

$$g_{11}\square^1v^1 + g_{12}\square^1v^2 + g_{21}\square^2v^1 + g_{22}\square^2v^2 \quad \square\begin{bmatrix}g_{11}v^1 + g_{12}v^2\\ g_{21}v^1 + g_{22}v^2\end{bmatrix}$$

$$\square^1\left(g_{11}v^1 + g_{12}v^2\right) + \square^2\left(g_{21}v^1 + g_{22}v^2\right) \quad \square\begin{bmatrix}g_{11}v^1 + g_{12}v^2\\ g_{21}v^1 + g_{22}v^2\end{bmatrix}$$

$$\square^1\left(g_{11}v^1 + g_{12}v^2\right) + \square^2\left(g_{21}v^1 + g_{22}v^2\right) = \begin{bmatrix}\square^1 & \square^2\end{bmatrix}\begin{bmatrix}g_{11}v^1 + g_{12}v^2\\ g_{21}v^1 + g_{22}v^2\end{bmatrix}$$

$$\mathfrak{g}\begin{bmatrix}g_{11} & g_{12}\\ g_{21} & g_{22}\end{bmatrix}\begin{bmatrix}v^1\\ v^2\end{bmatrix} \stackrel{\text{should}}{=} \begin{bmatrix}g_{11} & g_{12}\\ g_{21} & g_{22}\end{bmatrix}\begin{bmatrix}v^1\\ v^2\end{bmatrix}\mathfrak{g}$$

$$= \begin{bmatrix}\mathfrak{g}^{11} & \mathfrak{g}^{12}\\ \mathfrak{g}^{21} & \mathfrak{g}^{22}\end{bmatrix}\begin{bmatrix}g_{11} & g_{12}\\ g_{21} & g_{22}\end{bmatrix}\begin{bmatrix}v^1\\ v^2\end{bmatrix}$$

|cannot do matrix multiplication ⟻

$$= \begin{bmatrix}\mathfrak{g}^{11}g_{11} + \mathfrak{g}^{12}g_{21} & \mathfrak{g}^{11}g_{12} + \mathfrak{g}^{12}g_{22}\\ \mathfrak{g}^{21}g_{11} + \mathfrak{g}^{22}g_{21} & \mathfrak{g}^{21}g_{12} + \mathfrak{g}^{22}g_{22}\end{bmatrix}\begin{bmatrix}v^1\\ v^2\end{bmatrix} \qquad = \begin{bmatrix}\mathfrak{g}^{1k}g_{k1} & \mathfrak{g}^{1k}g_{k2}\\ \mathfrak{g}^{2k}g_{k1} & \mathfrak{g}^{2k}g_{k2}\end{bmatrix}\begin{bmatrix}v^1\\ v^2\end{bmatrix}$$

$$= \begin{bmatrix}\delta^1{}_1 & \delta^1{}_2\\ \delta^2{}_1 & \delta^2{}_2\end{bmatrix}\begin{bmatrix}v^1\\ v^2\end{bmatrix} \qquad \mathfrak{g}^{ik}g_{kj} = \delta^i{}_j$$

$$= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}v^1\\ v^2\end{bmatrix} = \begin{bmatrix}v^1\\ v^2\end{bmatrix}$$

$$\mathfrak{g}^{ik} g_{kj} = \delta^i{}_j$$
$$\Rightarrow g_{kj}\mathfrak{g}^{ik} = (\delta^i{}_j)^\intercal$$
$$= g_{jk}\mathfrak{g}^{ki} = \delta_j{}^i$$

$g_{jk} = g_{kj}$

if and should $\mathfrak{g}^{ki} = \mathfrak{g}^{ik}$

$$\begin{bmatrix}\begin{bmatrix}g_{11} & g_{12}\end{bmatrix} & \begin{bmatrix}g_{21} & g_{22}\end{bmatrix}\end{bmatrix} \begin{bmatrix}u^1\\ u^2\end{bmatrix} \begin{bmatrix}v^1\\ v^2\end{bmatrix} \equiv \begin{bmatrix}u^1 & u^2\end{bmatrix} \begin{bmatrix}g_{11} & g_{12}\\ g_{21} & g_{22}\end{bmatrix} \begin{bmatrix}v^1\\ v^2\end{bmatrix}$$

$$\begin{bmatrix}\begin{bmatrix}g_{11} & g_{12}\end{bmatrix} & \begin{bmatrix}g_{21} & g_{22}\end{bmatrix}\end{bmatrix} \begin{bmatrix}u^1\\ u^2\end{bmatrix} \stackrel{?}{=} \begin{bmatrix}u^1 & u^2\end{bmatrix} \begin{bmatrix}g_{11} & g_{12}\\ g_{21} & g_{22}\end{bmatrix}$$

$$\begin{aligned}&\begin{bmatrix}\begin{bmatrix}g_{11} & g_{12}\end{bmatrix} & \begin{bmatrix}g_{21} & g_{22}\end{bmatrix}\end{bmatrix} \begin{bmatrix}u^1\\ u^2\end{bmatrix}\\ &= \begin{bmatrix}g_{11} & g_{12}\end{bmatrix} u^1 + \begin{bmatrix}g_{21} & g_{22}\end{bmatrix} u^2\\ &= \begin{bmatrix}g_{11}u^1 & g_{12}u^1\end{bmatrix} + \begin{bmatrix}g_{21}u^2 & g_{22}u^2\end{bmatrix}\\ &= \begin{bmatrix}g_{11}u^1 + g_{21}u^2 & g_{12}u^1 + g_{22}u^2\end{bmatrix}\end{aligned}$$

$$\begin{aligned}&\begin{bmatrix}u^1 & u^2\end{bmatrix} \begin{bmatrix}g_{11} & g_{12}\\ g_{21} & g_{22}\end{bmatrix}\\ &= \begin{bmatrix}\begin{bmatrix}u^1 & u^2\end{bmatrix}\begin{bmatrix}g_{11}\\ g_{21}\end{bmatrix} & \begin{bmatrix}u^1 & u^2\end{bmatrix}\begin{bmatrix}g_{12}\\ g_{22}\end{bmatrix}\end{bmatrix}\\ &= \begin{bmatrix}u^1 g_{11} + u^2 g_{21} & u^1 g_{12} + u^2 g_{22}\end{bmatrix}\\ &= \begin{bmatrix}g_{11}u^1 + g_{21}u^2 & g_{12}u^1 + g_{22}u^2\end{bmatrix}\end{aligned}$$

$$\begin{bmatrix}\begin{bmatrix}g_{11} & g_{12}\end{bmatrix} & \begin{bmatrix}g_{21} & g_{22}\end{bmatrix}\end{bmatrix} \begin{bmatrix}u^1\\ u^2\end{bmatrix} \begin{bmatrix}v^1\\ v^2\end{bmatrix} \equiv \begin{bmatrix}u^1 & u^2\end{bmatrix} \begin{bmatrix}g_{11} & g_{12}\\ g_{21} & g_{22}\end{bmatrix} \begin{bmatrix}v^1\\ v^2\end{bmatrix}$$

$$\begin{bmatrix}\begin{bmatrix}g_{11} & g_{12}\end{bmatrix} & \begin{bmatrix}g_{21} & g_{22}\end{bmatrix}\end{bmatrix} \begin{bmatrix}u^1\\ u^2\end{bmatrix} \equiv \begin{bmatrix}u^1 & u^2\end{bmatrix} \begin{bmatrix}g_{11} & g_{12}\\ g_{21} & g_{22}\end{bmatrix}$$

$$\mathfrak{g}\begin{bmatrix}\begin{bmatrix}g_{11} & g_{12}\end{bmatrix} & \begin{bmatrix}g_{21} & g_{22}\end{bmatrix}\end{bmatrix} \begin{bmatrix}u^1\\ u^2\end{bmatrix} \stackrel{?}{=} \mathfrak{g}\begin{bmatrix}u^1 & u^2\end{bmatrix} \begin{bmatrix}g_{11} & g_{12}\\ g_{21} & g_{22}\end{bmatrix}$$

$$\mathfrak{g}\begin{bmatrix}u^1 & u^2\end{bmatrix} \begin{bmatrix}g_{11} & g_{12}\\ g_{21} & g_{22}\end{bmatrix} \stackrel{\text{should}}{=} \begin{bmatrix}u^1 & u^2\end{bmatrix} \begin{bmatrix}g_{11} & g_{12}\\ g_{21} & g_{22}\end{bmatrix}\mathfrak{g}$$

$$= \begin{bmatrix}\mathfrak{g}^{11} & \mathfrak{g}^{12}\\ \mathfrak{g}^{21} & \mathfrak{g}^{22}\end{bmatrix} \begin{bmatrix}u^1 & u^2\end{bmatrix} \begin{bmatrix}g_{11} & g_{12}\\ g_{21} & g_{22}\end{bmatrix} \qquad \mathfrak{g} = \begin{bmatrix}\mathfrak{g}^{11} & \mathfrak{g}^{12}\\ \mathfrak{g}^{21} & \mathfrak{g}^{22}\end{bmatrix}$$

=cannot do matrix multiplication

$$\begin{aligned}&\begin{bmatrix}u^1 & u^2\end{bmatrix} \begin{bmatrix}g_{11} & g_{12}\\ g_{21} & g_{22}\end{bmatrix}\mathfrak{g}\\ &= \begin{bmatrix}u^1 & u^2\end{bmatrix} \begin{bmatrix}g_{11} & g_{12}\\ g_{21} & g_{22}\end{bmatrix} \begin{bmatrix}\mathfrak{g}^{11} & \mathfrak{g}^{12}\\ \mathfrak{g}^{21} & \mathfrak{g}^{22}\end{bmatrix}\\ &= \begin{bmatrix}u^1 & u^2\end{bmatrix} \begin{bmatrix}g_{11}\mathfrak{g}^{11} + g_{12}\mathfrak{g}^{21} & g_{11}\mathfrak{g}^{12} + g_{12}\mathfrak{g}^{22}\\ g_{21}\mathfrak{g}^{11} + g_{22}\mathfrak{g}^{21} & g_{21}\mathfrak{g}^{12} + g_{22}\mathfrak{g}^{22}\end{bmatrix} \qquad = \begin{bmatrix}u^1 & u^2\end{bmatrix} \begin{bmatrix}g_{1k}\mathfrak{g}^{k1} & g_{1k}\mathfrak{g}^{k2}\\ g_{2k}\mathfrak{g}^{k1} & g_{2k}\mathfrak{g}^{k2}\end{bmatrix}\end{aligned}$$

$$g_{jk} = g_{kj}$$
$$g_{jk}\mathfrak{g}^{ki} = \delta_j{}^i \quad \overset{\mathfrak{g}^{ki} = \mathfrak{g}^{ik}}{\Leftarrow} \quad \mathfrak{g}^{ik} g_{kj} = \delta^i{}_j$$

$$\begin{aligned}&= \begin{bmatrix}u^1 & u^2\end{bmatrix} \begin{bmatrix}\delta_1{}^1 & \delta_1{}^2\\ \delta_2{}^1 & \delta_2{}^2\end{bmatrix}\\ &= \begin{bmatrix}u^1 & u^2\end{bmatrix} \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix}u^1 & u^2\end{bmatrix}\end{aligned}$$

$$\vec{v} = \begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix} \begin{bmatrix}v^1\\ v^2\end{bmatrix} = \vec{e}_1 v^1 + \vec{e}_2 v^2 = \begin{bmatrix}\tilde{\vec{e}}_1 & \tilde{\vec{e}}_2\end{bmatrix} \begin{bmatrix}\tilde{v}^1\\ \tilde{v}^2\end{bmatrix} = \tilde{\vec{e}}_1\tilde{v}^1 + \tilde{\vec{e}}_2\tilde{v}^2$$

$$\vec{v} = \begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix} \begin{bmatrix}v^1\\ v^2\end{bmatrix} = \vec{e}_1 v^1 + \vec{e}_2 v^2 = \begin{bmatrix}\tilde{\vec{e}}_1 & \tilde{\vec{e}}_2\end{bmatrix} \begin{bmatrix}\tilde{v}^1\\ \tilde{v}^2\end{bmatrix} = \tilde{\vec{e}}_1\tilde{v}^1 + \tilde{\vec{e}}_2\tilde{v}^2$$

$$\downarrow ?$$

$$\overleftarrow{v} = \begin{bmatrix}v^1 & v^2\end{bmatrix} \begin{bmatrix}\overleftarrow{\epsilon}^1\\ \overleftarrow{\epsilon}^2\end{bmatrix} = v^1\overleftarrow{\epsilon}^1 + v^2\overleftarrow{\epsilon}^2 = \begin{bmatrix}\tilde{v}^1 & \tilde{v}^2\end{bmatrix} \begin{bmatrix}\tilde{\overleftarrow{\epsilon}}^1\\ \tilde{\overleftarrow{\epsilon}}^2\end{bmatrix} = \tilde{v}^1\tilde{\overleftarrow{\epsilon}}^1 + \tilde{v}^2\tilde{\overleftarrow{\epsilon}}^2$$

(0,1)-tensor covariant basis vector	(1,0)-tensor contravariant vector component scalar	
$\begin{cases} \widetilde{\vec{e}}_j = \vec{e}_i \widetilde{e}^i{}_j \\ \vec{e}_j = \widetilde{\vec{e}}_i e^i{}_j \end{cases}$	$\begin{cases} \widetilde{v}^i = e^i{}_j v^j \\ v^i = \widetilde{e}^i{}_j \widetilde{v}^j \end{cases}$	(1,1)-tensor
covector component scalar	dual basis covector	linear map
$\begin{cases} \widetilde{\alpha}_j = \alpha_i \widetilde{e}^i{}_j \\ \alpha_j = \widetilde{\alpha}_i e^i{}_j \end{cases}$	$\begin{cases} \widetilde{\overleftarrow{\epsilon}}^i = e^i{}_j \overleftarrow{\epsilon}^j \\ \overleftarrow{\epsilon}^i = \widetilde{e}^i{}_j \widetilde{\overleftarrow{\epsilon}}^j \end{cases}$	$\begin{cases} \widetilde{L}^i{}_j = e^i{}_h L^h{}_k \widetilde{e}^k{}_j \\ L^i{}_j = \widetilde{e}^i{}_h \widetilde{L}^h{}_k e^k{}_j \end{cases}$
(0,2)-tensor bilinear form = 2-form: including metric tensor		
$\begin{cases} \widetilde{b}_{ij} = b_{hk} \widetilde{e}^h{}_i \widetilde{e}^k{}_j \\ b_{ij} = \widetilde{b}_{hk} e^h{}_i e^k{}_j \end{cases}$		

$$\begin{bmatrix} \widetilde{\vec{e}}_1 & \widetilde{\vec{e}}_2 \end{bmatrix} = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix} \begin{bmatrix} \widetilde{e}^1{}_1 & \widetilde{e}^1{}_2 \\ \widetilde{e}^2{}_1 & \widetilde{e}^2{}_2 \end{bmatrix} \quad \begin{bmatrix} \widetilde{\vec{e}}_1 \cdot \\ \widetilde{\vec{e}}_2 \cdot \end{bmatrix} = \begin{bmatrix} \widetilde{e}^1{}_1 & \widetilde{e}^1{}_2 \\ \widetilde{e}^2{}_1 & \widetilde{e}^2{}_2 \end{bmatrix}^\intercal \begin{bmatrix} \vec{e}_1 \cdot \\ \vec{e}_2 \cdot \end{bmatrix}$$

$$\begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix} = \begin{bmatrix} \widetilde{\vec{e}}_1 & \widetilde{\vec{e}}_2 \end{bmatrix} \begin{bmatrix} e^1{}_1 & e^1{}_2 \\ e^2{}_1 & e^2{}_2 \end{bmatrix} \quad \begin{bmatrix} \vec{e}_1 \cdot \\ \vec{e}_2 \cdot \end{bmatrix} = \begin{bmatrix} e^1{}_1 & e^1{}_2 \\ e^2{}_1 & e^2{}_2 \end{bmatrix}^\intercal \begin{bmatrix} \widetilde{\vec{e}}_1 \cdot \\ \widetilde{\vec{e}}_2 \cdot \end{bmatrix}$$

Table 1.3: transformation as ordered tensor

$$\vec{v} = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \vec{e}_1 v^1 + \vec{e}_2 v^2 = \begin{bmatrix} \widetilde{\vec{e}}_1 & \widetilde{\vec{e}}_2 \end{bmatrix} \begin{bmatrix} \widetilde{v}^1 \\ \widetilde{v}^2 \end{bmatrix} = \widetilde{\vec{e}}_1 \widetilde{v}^1 + \widetilde{\vec{e}}_2 \widetilde{v}^2$$

$$\begin{cases} \vec{e}_j = \widetilde{\vec{e}}_i e^i{}_j \\ v^i = \widetilde{e}^i{}_j \widetilde{v}^j \end{cases} \overset{}{=} \begin{bmatrix} \widetilde{\vec{e}}_1 & \widetilde{\vec{e}}_2 \end{bmatrix} \begin{bmatrix} e^1{}_1 & e^1{}_2 \\ e^2{}_1 & e^2{}_2 \end{bmatrix} \begin{bmatrix} \widetilde{e}^1{}_1 & \widetilde{e}^1{}_2 \\ \widetilde{e}^2{}_1 & \widetilde{e}^2{}_2 \end{bmatrix} \begin{bmatrix} \widetilde{v}^1 \\ \widetilde{v}^2 \end{bmatrix} \quad \begin{cases} \widetilde{\vec{e}}_j = \vec{e}_i \widetilde{e}^i{}_j \\ \widetilde{v}^i = e^i{}_j v^j \end{cases} = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix} \begin{bmatrix} \widetilde{e}^1{}_1 & \widetilde{e}^1{}_2 \\ \widetilde{e}^2{}_1 & \widetilde{e}^2{}_2 \end{bmatrix} \begin{bmatrix} e^1{}_1 & e^1{}_2 \\ e^2{}_1 & e^2{}_2 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix}$$

$$= \begin{bmatrix} \widetilde{\vec{e}}_1 & \widetilde{\vec{e}}_2 \end{bmatrix} \begin{bmatrix} \widetilde{v}^1 \\ \widetilde{v}^2 \end{bmatrix} = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix}$$

$$\downarrow ?$$

$$\overleftarrow{v} = \begin{bmatrix} v^1 & v^2 \end{bmatrix} \begin{bmatrix} \overleftarrow{\epsilon}^1 \\ \overleftarrow{\epsilon}^2 \end{bmatrix} = v^1 \overleftarrow{\epsilon}^1 + v^2 \overleftarrow{\epsilon}^2 \overset{?}{=} \begin{bmatrix} \widetilde{v}^1 & \widetilde{v}^2 \end{bmatrix} \begin{bmatrix} \widetilde{\overleftarrow{\epsilon}}^1 \\ \widetilde{\overleftarrow{\epsilon}}^2 \end{bmatrix} = \widetilde{v}^1 \widetilde{\overleftarrow{\epsilon}}^1 + \widetilde{v}^2 \widetilde{\overleftarrow{\epsilon}}^2$$

$$\begin{cases} \vec{e}_j = \widetilde{\vec{e}}_i e^i{}_j \\ v^i = \widetilde{e}^i{}_j \widetilde{v}^j \end{cases} = \begin{bmatrix} \widetilde{v}^1 & \widetilde{v}^2 \end{bmatrix} \begin{bmatrix} \widetilde{e}^1{}_1 & \widetilde{e}^1{}_2 \\ \widetilde{e}^2{}_1 & \widetilde{e}^2{}_2 \end{bmatrix}^\intercal \begin{bmatrix} \widetilde{e}^1{}_1 & \widetilde{e}^1{}_2 \\ \widetilde{e}^2{}_1 & \widetilde{e}^2{}_2 \end{bmatrix} \begin{bmatrix} \widetilde{\overleftarrow{\epsilon}}^1 \\ \widetilde{\overleftarrow{\epsilon}}^2 \end{bmatrix} \overset{?}{=} \begin{cases} \widetilde{\vec{e}}_j = \vec{e}_i \widetilde{e}^i{}_j \\ \widetilde{v}^i = e^i{}_j v^j \end{cases} = \begin{bmatrix} v^1 & v^2 \end{bmatrix} \begin{bmatrix} e^1{}_1 & e^1{}_2 \\ e^2{}_1 & e^2{}_2 \end{bmatrix}^\intercal \begin{bmatrix} e^1{}_1 & e^1{}_2 \\ e^2{}_1 & e^2{}_2 \end{bmatrix} \begin{bmatrix} \overleftarrow{\epsilon}^1 \\ \overleftarrow{\epsilon}^2 \end{bmatrix}$$

$\not\equiv \begin{bmatrix} \widetilde{v}^1 & \widetilde{v}^2 \end{bmatrix} \begin{bmatrix} \widetilde{\overleftarrow{\epsilon}}^1 \\ \widetilde{\overleftarrow{\epsilon}}^2 \end{bmatrix} \not\equiv \begin{bmatrix} v^1 & v^2 \end{bmatrix} \begin{bmatrix} \overleftarrow{\epsilon}^1 \\ \overleftarrow{\epsilon}^2 \end{bmatrix}$ so this kind of definition makes not invariant $\overleftarrow{v}$

$$\vec{v} = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \vec{e}_1 v^1 + \vec{e}_2 v^2 = \begin{bmatrix} \widetilde{\vec{e}}_1 & \widetilde{\vec{e}}_2 \end{bmatrix} \begin{bmatrix} \widetilde{v}^1 \\ \widetilde{v}^2 \end{bmatrix} = \widetilde{\vec{e}}_1 \widetilde{v}^1 + \widetilde{\vec{e}}_2 \widetilde{v}^2$$

$$\vec{u} = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} = \vec{e}_1 u^1 + \vec{e}_2 u^2 = \begin{bmatrix} \widetilde{\vec{e}}_1 & \widetilde{\vec{e}}_2 \end{bmatrix} \begin{bmatrix} \widetilde{u}^1 \\ \widetilde{u}^2 \end{bmatrix} = \widetilde{\vec{e}}_1 \widetilde{u}^1 + \widetilde{\vec{e}}_2 \widetilde{u}^2$$

$$g = \begin{bmatrix} \vec{e}_1 \cdot \vec{e}_1 & \vec{e}_1 \cdot \vec{e}_2 \\ \vec{e}_2 \cdot \vec{e}_1 & \vec{e}_2 \cdot \vec{e}_2 \end{bmatrix} \Leftrightarrow g_{ij} = \vec{e}_i \cdot \vec{e}_j$$

$$\widetilde{g} = \begin{bmatrix} \widetilde{\vec{e}}_1 \cdot \widetilde{\vec{e}}_1 & \widetilde{\vec{e}}_1 \cdot \widetilde{\vec{e}}_2 \\ \widetilde{\vec{e}}_2 \cdot \widetilde{\vec{e}}_1 & \widetilde{\vec{e}}_2 \cdot \widetilde{\vec{e}}_2 \end{bmatrix} \Leftrightarrow \widetilde{g}_{ij} = \widetilde{\vec{e}}_i \cdot \widetilde{\vec{e}}_j$$

$$
\begin{aligned}
&\vec{u}\cdot\vec{v} = \vec{u}^{\intercal} g \vec{v} \\
=&\begin{bmatrix} u^1 & u^2 \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \begin{bmatrix} \tilde{u}^1 & \tilde{u}^2 \end{bmatrix} \begin{bmatrix} \tilde{g}_{11} & \tilde{g}_{12} \\ \tilde{g}_{21} & \tilde{g}_{22} \end{bmatrix} \begin{bmatrix} \tilde{v}^1 \\ \tilde{v}^2 \end{bmatrix} \\
=&\begin{bmatrix} u^1 & u^2 \end{bmatrix} \begin{bmatrix} \vec{e}_1\cdot\vec{e}_1 & \vec{e}_1\cdot\vec{e}_2 \\ \vec{e}_2\cdot\vec{e}_1 & \vec{e}_2\cdot\vec{e}_2 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \begin{bmatrix} u^1 & u^2 \end{bmatrix} \begin{bmatrix} \vec{e}_1\cdot \\ \vec{e}_2\cdot \end{bmatrix} \begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \\
=&\left(u^1\vec{e}_1\cdot + u^2\vec{e}_2\cdot\right)\left(\vec{e}_1 v^1 + \vec{e}_2 v^2\right) \\
=&u^1\vec{e}_1\cdot\left(\vec{e}_1 v^1 + \vec{e}_2 v^2\right) + u^2\vec{e}_2\cdot\left(\vec{e}_1 v^1 + \vec{e}_2 v^2\right) \\
=&\left(u^1\vec{e}_1\cdot\vec{e}_1 v^1 + u^1\vec{e}_1\cdot\vec{e}_2 v^2\right) + \left(u^2\vec{e}_2\cdot\vec{e}_1 v^1 + u^2\vec{e}_2\cdot\vec{e}_2 v^2\right) \\
=&\begin{bmatrix} \tilde{u}^1 & \tilde{u}^2 \end{bmatrix} \begin{bmatrix} \tilde{\vec{e}}_1\cdot\tilde{\vec{e}}_1 & \tilde{\vec{e}}_1\cdot\tilde{\vec{e}}_2 \\ \tilde{\vec{e}}_2\cdot\tilde{\vec{e}}_1 & \tilde{\vec{e}}_2\cdot\tilde{\vec{e}}_2 \end{bmatrix} \begin{bmatrix} \tilde{v}^1 \\ \tilde{v}^2 \end{bmatrix} = \begin{bmatrix} \tilde{u}^1 & \tilde{u}^2 \end{bmatrix} \begin{bmatrix} \tilde{\vec{e}}_1\cdot \\ \tilde{\vec{e}}_2\cdot \end{bmatrix} \begin{bmatrix} \tilde{\vec{e}}_1 & \tilde{\vec{e}}_2 \end{bmatrix} \begin{bmatrix} \tilde{v}^1 \\ \tilde{v}^2 \end{bmatrix} \\
=&\left(\tilde{u}^1\tilde{\vec{e}}_1\cdot + \tilde{u}^2\tilde{\vec{e}}_2\cdot\right)\left(\tilde{\vec{e}}_1\tilde{v}^1 + \tilde{\vec{e}}_2\tilde{v}^2\right) \\
=&\tilde{u}^1\tilde{\vec{e}}_1\cdot\left(\tilde{\vec{e}}_1\tilde{v}^1 + \tilde{\vec{e}}_2\tilde{v}^2\right) + \tilde{u}^2\tilde{\vec{e}}_2\cdot\left(\tilde{\vec{e}}_1\tilde{v}^1 + \tilde{\vec{e}}_2\tilde{v}^2\right) \\
=&\left(\tilde{u}^1\tilde{\vec{e}}_1\cdot\tilde{\vec{e}}_1\tilde{v}^1 + \tilde{u}^1\tilde{\vec{e}}_1\cdot\tilde{\vec{e}}_2\tilde{v}^2\right) + \left(\tilde{u}^2\tilde{\vec{e}}_2\cdot\tilde{\vec{e}}_1\tilde{v}^1 + \tilde{u}^2\tilde{\vec{e}}_2\cdot\tilde{\vec{e}}_2\tilde{v}^2\right)
\end{aligned}
$$

$$
\begin{aligned}
&\vec{v}\cdot\vec{v} = \vec{v}^{\intercal} g \vec{v} \\
=&\begin{bmatrix} v^1 & v^2 \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \begin{bmatrix} \tilde{v}^1 & \tilde{v}^2 \end{bmatrix} \begin{bmatrix} \tilde{g}_{11} & \tilde{g}_{12} \\ \tilde{g}_{21} & \tilde{g}_{22} \end{bmatrix} \begin{bmatrix} \tilde{v}^1 \\ \tilde{v}^2 \end{bmatrix} \\
=&\begin{bmatrix} v^1 & v^2 \end{bmatrix} \begin{bmatrix} \vec{e}_1\cdot\vec{e}_1 & \vec{e}_1\cdot\vec{e}_2 \\ \vec{e}_2\cdot\vec{e}_1 & \vec{e}_2\cdot\vec{e}_2 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \begin{bmatrix} v^1 & v^2 \end{bmatrix} \begin{bmatrix} \vec{e}_1\cdot \\ \vec{e}_2\cdot \end{bmatrix} \begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \\
=&\left(v^1\vec{e}_1\cdot + v^2\vec{e}_2\cdot\right)\left(\vec{e}_1 v^1 + \vec{e}_2 v^2\right) \\
=&v^1\vec{e}_1\cdot\left(\vec{e}_1 v^1 + \vec{e}_2 v^2\right) + v^2\vec{e}_2\cdot\left(\vec{e}_1 v^1 + \vec{e}_2 v^2\right) \\
=&\left(v^1\vec{e}_1\cdot\vec{e}_1 v^1 + v^1\vec{e}_1\cdot\vec{e}_2 v^2\right) + \left(v^2\vec{e}_2\cdot\vec{e}_1 v^1 + v^2\vec{e}_2\cdot\vec{e}_2 v^2\right) \\
=&\begin{bmatrix} \tilde{v}^1 & \tilde{v}^2 \end{bmatrix} \begin{bmatrix} \tilde{\vec{e}}_1\cdot\tilde{\vec{e}}_1 & \tilde{\vec{e}}_1\cdot\tilde{\vec{e}}_2 \\ \tilde{\vec{e}}_2\cdot\tilde{\vec{e}}_1 & \tilde{\vec{e}}_2\cdot\tilde{\vec{e}}_2 \end{bmatrix} \begin{bmatrix} \tilde{v}^1 \\ \tilde{v}^2 \end{bmatrix} = \begin{bmatrix} \tilde{v}^1 & \tilde{v}^2 \end{bmatrix} \begin{bmatrix} \tilde{\vec{e}}_1\cdot \\ \tilde{\vec{e}}_2\cdot \end{bmatrix} \begin{bmatrix} \tilde{\vec{e}}_1 & \tilde{\vec{e}}_2 \end{bmatrix} \begin{bmatrix} \tilde{v}^1 \\ \tilde{v}^2 \end{bmatrix} \\
=&\left(\tilde{v}^1\tilde{\vec{e}}_1\cdot + \tilde{v}^2\tilde{\vec{e}}_2\cdot\right)\left(\tilde{\vec{e}}_1\tilde{v}^1 + \tilde{\vec{e}}_2\tilde{v}^2\right) \\
=&\tilde{v}^1\tilde{\vec{e}}_1\cdot\left(\tilde{\vec{e}}_1\tilde{v}^1 + \tilde{\vec{e}}_2\tilde{v}^2\right) + \tilde{v}^2\tilde{\vec{e}}_2\cdot\left(\tilde{\vec{e}}_1\tilde{v}^1 + \tilde{\vec{e}}_2\tilde{v}^2\right) \\
=&\left(\tilde{v}^1\tilde{\vec{e}}_1\cdot\tilde{\vec{e}}_1\tilde{v}^1 + \tilde{v}^1\tilde{\vec{e}}_1\cdot\tilde{\vec{e}}_2\tilde{v}^2\right) + \left(\tilde{v}^2\tilde{\vec{e}}_2\cdot\tilde{\vec{e}}_1\tilde{v}^1 + \tilde{v}^2\tilde{\vec{e}}_2\cdot\tilde{\vec{e}}_2\tilde{v}^2\right)
\end{aligned}
$$

$$\vec{v} = \begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\begin{bmatrix}v^1\\ v^2\end{bmatrix} = \vec{e}_1 v^1 + \vec{e}_2 v^2 = \begin{bmatrix}\tilde{\vec{e}}_1 & \tilde{\vec{e}}_2\end{bmatrix}\begin{bmatrix}\tilde{v}^1\\ \tilde{v}^2\end{bmatrix} = \tilde{\vec{e}}_1 \tilde{v}^1 + \tilde{\vec{e}}_2 \tilde{v}^2$$

$$\begin{cases}\vec{e}_j = \tilde{\vec{e}}_i e^i{}_j\\ v^i = \tilde{e}^i{}_j \tilde{v}^j\end{cases} = \begin{bmatrix}\tilde{\vec{e}}_1 & \tilde{\vec{e}}_2\end{bmatrix}\begin{bmatrix}e^1{}_1 & e^1{}_2\\ e^2{}_1 & e^2{}_2\end{bmatrix}\begin{bmatrix}\tilde{e}^1{}_1 & \tilde{e}^1{}_2\\ \tilde{e}^2{}_1 & \tilde{e}^2{}_2\end{bmatrix}\begin{bmatrix}\tilde{v}^1\\ \tilde{v}^2\end{bmatrix} \qquad \begin{cases}\tilde{\vec{e}}_j = \vec{e}_i \tilde{e}^i{}_j\\ \tilde{v}^i = e^i{}_j v^j\end{cases} = \begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\begin{bmatrix}\tilde{e}^1{}_1 & \tilde{e}^1{}_2\\ \tilde{e}^2{}_1 & \tilde{e}^2{}_2\end{bmatrix}\begin{bmatrix}e^1{}_1 & e^1{}_2\\ e^2{}_1 & e^2{}_2\end{bmatrix}\begin{bmatrix}v^1\\ v^2\end{bmatrix}$$

$$= \begin{bmatrix}\tilde{\vec{e}}_1 & \tilde{\vec{e}}_2\end{bmatrix}\begin{bmatrix}\tilde{v}^1\\ \tilde{v}^2\end{bmatrix} = \begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\begin{bmatrix}v^1\\ v^2\end{bmatrix}$$

$$\downarrow ?$$

$$\overleftarrow{v} = \vec{v}\cdot\square = \begin{bmatrix}v^1 & v^2\end{bmatrix}\begin{bmatrix}g_{11} & g_{12}\\ g_{21} & g_{22}\end{bmatrix} = \begin{bmatrix}\tilde{v}^1 & \tilde{v}^2\end{bmatrix}\begin{bmatrix}\tilde{g}_{11} & \tilde{g}_{12}\\ \tilde{g}_{21} & \tilde{g}_{22}\end{bmatrix}$$

$$\leftrightarrow \begin{bmatrix}v^1 & v^2\end{bmatrix}\begin{bmatrix}\vec{e}_1\cdot\vec{e}_1 & \vec{e}_1\cdot\vec{e}_2\\ \vec{e}_2\cdot\vec{e}_1 & \vec{e}_2\cdot\vec{e}_2\end{bmatrix} = \begin{bmatrix}\tilde{v}^1 & \tilde{v}^2\end{bmatrix}\begin{bmatrix}\tilde{\vec{e}}_1\cdot\tilde{\vec{e}}_1 & \tilde{\vec{e}}_1\cdot\tilde{\vec{e}}_2\\ \tilde{\vec{e}}_2\cdot\tilde{\vec{e}}_1 & \tilde{\vec{e}}_2\cdot\tilde{\vec{e}}_2\end{bmatrix}$$

$$= \begin{bmatrix}v^1 & v^2\end{bmatrix}\begin{bmatrix}\vec{e}_1\cdot\\ \vec{e}_2\cdot\end{bmatrix}\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix} = \begin{bmatrix}\tilde{v}^1 & \tilde{v}^2\end{bmatrix}\begin{bmatrix}\tilde{\vec{e}}_1\cdot\\ \tilde{\vec{e}}_2\cdot\end{bmatrix}\begin{bmatrix}\tilde{\vec{e}}_1 & \tilde{\vec{e}}_2\end{bmatrix}$$

$$\begin{bmatrix}\tilde{\vec{e}}_1 & \tilde{\vec{e}}_2\end{bmatrix} = \begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\begin{bmatrix}\tilde{e}^1{}_1 & \tilde{e}^1{}_2\\ \tilde{e}^2{}_1 & \tilde{e}^2{}_2\end{bmatrix} \qquad \begin{bmatrix}\tilde{\vec{e}}_1\cdot\\ \tilde{\vec{e}}_2\cdot\end{bmatrix} = \begin{bmatrix}\tilde{e}^1{}_1 & \tilde{e}^1{}_2\\ \tilde{e}^2{}_1 & \tilde{e}^2{}_2\end{bmatrix}^\intercal\begin{bmatrix}\vec{e}_1\cdot\\ \vec{e}_2\cdot\end{bmatrix}$$

$$\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix} = \begin{bmatrix}\tilde{\vec{e}}_1 & \tilde{\vec{e}}_2\end{bmatrix}\begin{bmatrix}e^1{}_1 & e^1{}_2\\ e^2{}_1 & e^2{}_2\end{bmatrix} \qquad \begin{bmatrix}\vec{e}_1\cdot\\ \vec{e}_2\cdot\end{bmatrix} = \begin{bmatrix}e^1{}_1 & e^1{}_2\\ e^2{}_1 & e^2{}_2\end{bmatrix}^\intercal\begin{bmatrix}\tilde{\vec{e}}_1\cdot\\ \tilde{\vec{e}}_2\cdot\end{bmatrix}$$

$$\overleftarrow{v} = \vec{v}\cdot\square = \begin{bmatrix}v^1 & v^2\end{bmatrix}\begin{bmatrix}g_{11} & g_{12}\\ g_{21} & g_{22}\end{bmatrix} = \begin{bmatrix}\tilde{v}^1 & \tilde{v}^2\end{bmatrix}\begin{bmatrix}\tilde{g}_{11} & \tilde{g}_{12}\\ \tilde{g}_{21} & \tilde{g}_{22}\end{bmatrix}$$

$$\leftrightarrow \begin{bmatrix}v^1 & v^2\end{bmatrix}\begin{bmatrix}\vec{e}_1\cdot\vec{e}_1 & \vec{e}_1\cdot\vec{e}_2\\ \vec{e}_2\cdot\vec{e}_1 & \vec{e}_2\cdot\vec{e}_2\end{bmatrix}\begin{bmatrix}\overleftarrow{\epsilon}^1\\ \overleftarrow{\epsilon}^2\end{bmatrix} \stackrel{?}{=} \begin{bmatrix}\tilde{v}^1 & \tilde{v}^2\end{bmatrix}\begin{bmatrix}\tilde{\vec{e}}_1\cdot\tilde{\vec{e}}_1 & \tilde{\vec{e}}_1\cdot\tilde{\vec{e}}_2\\ \tilde{\vec{e}}_2\cdot\tilde{\vec{e}}_1 & \tilde{\vec{e}}_2\cdot\tilde{\vec{e}}_2\end{bmatrix}\begin{bmatrix}\tilde{\overleftarrow{\epsilon}}^1\\ \tilde{\overleftarrow{\epsilon}}^2\end{bmatrix}$$

$$= \begin{bmatrix}v^1 & v^2\end{bmatrix}\begin{bmatrix}\vec{e}_1\cdot\\ \vec{e}_2\cdot\end{bmatrix}\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\begin{bmatrix}\overleftarrow{\epsilon}^1\\ \overleftarrow{\epsilon}^2\end{bmatrix} \stackrel{?}{=} \begin{bmatrix}\tilde{v}^1 & \tilde{v}^2\end{bmatrix}\begin{bmatrix}\tilde{\vec{e}}_1\cdot\\ \tilde{\vec{e}}_2\cdot\end{bmatrix}\begin{bmatrix}\tilde{\vec{e}}_1 & \tilde{\vec{e}}_2\end{bmatrix}\begin{bmatrix}\tilde{\overleftarrow{\epsilon}}^1\\ \tilde{\overleftarrow{\epsilon}}^2\end{bmatrix}$$

$$= \begin{bmatrix}\tilde{v}^1 & \tilde{v}^2\end{bmatrix}\begin{bmatrix}\tilde{e}^1{}_1 & \tilde{e}^1{}_2\\ \tilde{e}^2{}_1 & \tilde{e}^2{}_2\end{bmatrix}^\intercal\begin{bmatrix}e^1{}_1 & e^1{}_2\\ e^2{}_1 & e^2{}_2\end{bmatrix}^\intercal\begin{bmatrix}\tilde{\vec{e}}_1\cdot\\ \tilde{\vec{e}}_2\cdot\end{bmatrix}\begin{bmatrix}\tilde{\vec{e}}_1 & \tilde{\vec{e}}_2\end{bmatrix}\begin{bmatrix}e^1{}_1 & e^1{}_2\\ e^2{}_1 & e^2{}_2\end{bmatrix}\begin{bmatrix}\tilde{e}^1{}_1 & \tilde{e}^1{}_2\\ \tilde{e}^2{}_1 & \tilde{e}^2{}_2\end{bmatrix}\begin{bmatrix}\tilde{\overleftarrow{\epsilon}}^1\\ \tilde{\overleftarrow{\epsilon}}^2\end{bmatrix}$$

$$= \begin{bmatrix}v^1 & v^2\end{bmatrix}\begin{bmatrix}e^1{}_1 & e^1{}_2\\ e^2{}_1 & e^2{}_2\end{bmatrix}^\intercal\begin{bmatrix}\tilde{e}^1{}_1 & \tilde{e}^1{}_2\\ \tilde{e}^2{}_1 & \tilde{e}^2{}_2\end{bmatrix}^\intercal\begin{bmatrix}\vec{e}_1\cdot\\ \vec{e}_2\cdot\end{bmatrix}\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\begin{bmatrix}\tilde{e}^1{}_1 & \tilde{e}^1{}_2\\ \tilde{e}^2{}_1 & \tilde{e}^2{}_2\end{bmatrix}\begin{bmatrix}e^1{}_1 & e^1{}_2\\ e^2{}_1 & e^2{}_2\end{bmatrix}\begin{bmatrix}\overleftarrow{\epsilon}^1\\ \overleftarrow{\epsilon}^2\end{bmatrix}$$

$$= \begin{bmatrix}\tilde{v}^1 & \tilde{v}^2\end{bmatrix}\left(\begin{bmatrix}e^1{}_1 & e^1{}_2\\ e^2{}_1 & e^2{}_2\end{bmatrix}\begin{bmatrix}\tilde{e}^1{}_1 & \tilde{e}^1{}_2\\ \tilde{e}^2{}_1 & \tilde{e}^2{}_2\end{bmatrix}\right)^\intercal\begin{bmatrix}\tilde{\vec{e}}_1\cdot\\ \tilde{\vec{e}}_2\cdot\end{bmatrix}\begin{bmatrix}\tilde{\vec{e}}_1 & \tilde{\vec{e}}_2\end{bmatrix}\left(\begin{bmatrix}e^1{}_1 & e^1{}_2\\ e^2{}_1 & e^2{}_2\end{bmatrix}\begin{bmatrix}\tilde{e}^1{}_1 & \tilde{e}^1{}_2\\ \tilde{e}^2{}_1 & \tilde{e}^2{}_2\end{bmatrix}\right)\begin{bmatrix}\tilde{\overleftarrow{\epsilon}}^1\\ \tilde{\overleftarrow{\epsilon}}^2\end{bmatrix}$$

$$= \begin{bmatrix}v^1 & v^2\end{bmatrix}\left(\begin{bmatrix}\tilde{e}^1{}_1 & \tilde{e}^1{}_2\\ \tilde{e}^2{}_1 & \tilde{e}^2{}_2\end{bmatrix}\begin{bmatrix}e^1{}_1 & e^1{}_2\\ e^2{}_1 & e^2{}_2\end{bmatrix}\right)^\intercal\begin{bmatrix}\vec{e}_1\cdot\\ \vec{e}_2\cdot\end{bmatrix}\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\left(\begin{bmatrix}\tilde{e}^1{}_1 & \tilde{e}^1{}_2\\ \tilde{e}^2{}_1 & \tilde{e}^2{}_2\end{bmatrix}\begin{bmatrix}e^1{}_1 & e^1{}_2\\ e^2{}_1 & e^2{}_2\end{bmatrix}\right)\begin{bmatrix}\overleftarrow{\epsilon}^1\\ \overleftarrow{\epsilon}^2\end{bmatrix}$$

$$= \begin{bmatrix}\tilde{v}^1 & \tilde{v}^2\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}^\intercal\begin{bmatrix}\tilde{\vec{e}}_1\cdot\\ \tilde{\vec{e}}_2\cdot\end{bmatrix}\begin{bmatrix}\tilde{\vec{e}}_1 & \tilde{\vec{e}}_2\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\tilde{\overleftarrow{\epsilon}}^1\\ \tilde{\overleftarrow{\epsilon}}^2\end{bmatrix} = \begin{bmatrix}\tilde{v}^1 & \tilde{v}^2\end{bmatrix}\begin{bmatrix}\tilde{\vec{e}}_1\cdot\\ \tilde{\vec{e}}_2\cdot\end{bmatrix}\begin{bmatrix}\tilde{\vec{e}}_1 & \tilde{\vec{e}}_2\end{bmatrix}\begin{bmatrix}\tilde{\overleftarrow{\epsilon}}^1\\ \tilde{\overleftarrow{\epsilon}}^2\end{bmatrix}$$

$$= \begin{bmatrix}v^1 & v^2\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}^\intercal\begin{bmatrix}\vec{e}_1\cdot\\ \vec{e}_2\cdot\end{bmatrix}\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\overleftarrow{\epsilon}^1\\ \overleftarrow{\epsilon}^2\end{bmatrix} = \begin{bmatrix}v^1 & v^2\end{bmatrix}\begin{bmatrix}\vec{e}_1\cdot\\ \vec{e}_2\cdot\end{bmatrix}\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\begin{bmatrix}\overleftarrow{\epsilon}^1\\ \overleftarrow{\epsilon}^2\end{bmatrix}$$

$$\text{invariant } \overleftarrow{v} = \vec{v}\cdot\square = \begin{bmatrix}v^1 & v^2\end{bmatrix}\begin{bmatrix}\vec{e}_1\cdot\\ \vec{e}_2\cdot\end{bmatrix}\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\begin{bmatrix}\overleftarrow{\epsilon}^1\\ \overleftarrow{\epsilon}^2\end{bmatrix} = \begin{bmatrix}\tilde{v}^1 & \tilde{v}^2\end{bmatrix}\begin{bmatrix}\tilde{\vec{e}}_1\cdot\\ \tilde{\vec{e}}_2\cdot\end{bmatrix}\begin{bmatrix}\tilde{\vec{e}}_1 & \tilde{\vec{e}}_2\end{bmatrix}\begin{bmatrix}\tilde{\overleftarrow{\epsilon}}^1\\ \tilde{\overleftarrow{\epsilon}}^2\end{bmatrix}$$

$$\begin{bmatrix} g_{11} & g_{12}\\ g_{21} & g_{22}\end{bmatrix} = \begin{bmatrix}\vec{e}_1\cdot\vec{e}_1 & \vec{e}_1\cdot\vec{e}_2\\ \vec{e}_2\cdot\vec{e}_1 & \vec{e}_2\cdot\vec{e}_2\end{bmatrix} = \begin{bmatrix}\vec{e}_1\cdot\\ \vec{e}_2\cdot\end{bmatrix}\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix} = \begin{bmatrix}e^1{}_1 & e^1{}_2\\ e^2{}_1 & e^2{}_2\end{bmatrix}^\top \begin{bmatrix}\tilde{\vec{e}}_1\cdot\\ \tilde{\vec{e}}_2\cdot\end{bmatrix}\begin{bmatrix}\tilde{\vec{e}}_1 & \tilde{\vec{e}}_2\end{bmatrix}\begin{bmatrix}e^1{}_1 & e^1{}_2\\ e^2{}_1 & e^2{}_2\end{bmatrix}$$

$$= \begin{bmatrix}e^1{}_1 & e^1{}_2\\ e^2{}_1 & e^2{}_2\end{bmatrix}^\top \begin{bmatrix}\tilde{\vec{e}}_1\cdot\tilde{\vec{e}}_1 & \tilde{\vec{e}}_1\cdot\tilde{\vec{e}}_2\\ \tilde{\vec{e}}_2\cdot\tilde{\vec{e}}_1 & \tilde{\vec{e}}_2\cdot\tilde{\vec{e}}_2\end{bmatrix}\begin{bmatrix}e^1{}_1 & e^1{}_2\\ e^2{}_1 & e^2{}_2\end{bmatrix} = \begin{bmatrix}e^1{}_1 & e^1{}_2\\ e^2{}_1 & e^2{}_2\end{bmatrix}^\top \begin{bmatrix}\widetilde{g}_{11} & \widetilde{g}_{12}\\ \widetilde{g}_{21} & \widetilde{g}_{22}\end{bmatrix}\begin{bmatrix}e^1{}_1 & e^1{}_2\\ e^2{}_1 & e^2{}_2\end{bmatrix}$$

$$= \begin{cases} \begin{bmatrix}e^1{}_1 & e^1{}_2\\ e^2{}_1 & e^2{}_2\end{bmatrix}^\top \begin{bmatrix}\widetilde{g}_{11} & \widetilde{g}_{12}\\ \widetilde{g}_{21} & \widetilde{g}_{22}\end{bmatrix}\begin{bmatrix}e^1{}_1 & e^1{}_2\\ e^2{}_1 & e^2{}_2\end{bmatrix} & \begin{bmatrix}e^1{}_1 & e^1{}_2\\ e^2{}_1 & e^2{}_2\end{bmatrix}^\top = \begin{bmatrix}e_1^{\top 1} & e_2^{\top 1}\\ e_1^{\top 2} & e_2^{\top 2}\end{bmatrix} \\ \begin{bmatrix}e^1{}_1 & e^1{}_2\\ e^2{}_1 & e^2{}_2\end{bmatrix}^\top \begin{bmatrix}\widetilde{g}_{11} & \widetilde{g}_{12}\\ \widetilde{g}_{21} & \widetilde{g}_{22}\end{bmatrix}\begin{bmatrix}e^1{}_1 & e^1{}_2\\ e^2{}_1 & e^2{}_2\end{bmatrix} & \begin{bmatrix}e^1{}_1 & e^1{}_2\\ e^2{}_1 & e^2{}_2\end{bmatrix}^\top = \begin{bmatrix}e_1^{\top 1} & e_1^{\top 2}\\ e_2^{\top 1} & e_2^{\top 2}\end{bmatrix} \end{cases}$$

$$= \begin{cases} \begin{bmatrix}e_1^{\top 1} & e_2^{\top 1}\\ e_1^{\top 2} & e_2^{\top 2}\end{bmatrix}\begin{bmatrix}\widetilde{g}_{11} & \widetilde{g}_{12}\\ \widetilde{g}_{21} & \widetilde{g}_{22}\end{bmatrix}\begin{bmatrix}e^1{}_1 & e^1{}_2\\ e^2{}_1 & e^2{}_2\end{bmatrix} = \begin{bmatrix}e_1^{\top 1}\widetilde{g}_{11} + e_2^{\top 1}\widetilde{g}_{21} & e_1^{\top 1}\widetilde{g}_{12} + e_2^{\top 1}\widetilde{g}_{22}\\ e_1^{\top 2}\widetilde{g}_{11} + e_2^{\top 2}\widetilde{g}_{21} & e_1^{\top 2}\widetilde{g}_{12} + e_2^{\top 2}\widetilde{g}_{22}\end{bmatrix}\begin{bmatrix}e^1{}_1 & e^1{}_2\\ e^2{}_1 & e^2{}_2\end{bmatrix} \\ \begin{bmatrix}e_1^{\top 1} & e_1^{\top 2}\\ e_2^{\top 1} & e_2^{\top 2}\end{bmatrix}\begin{bmatrix}\widetilde{g}_{11} & \widetilde{g}_{12}\\ \widetilde{g}_{21} & \widetilde{g}_{22}\end{bmatrix}\begin{bmatrix}e^1{}_1 & e^1{}_2\\ e^2{}_1 & e^2{}_2\end{bmatrix} = \begin{bmatrix}e_1^{\top 1}\widetilde{g}_{11} + e_1^{\top 2}\widetilde{g}_{21} & e_1^{\top 1}\widetilde{g}_{12} + e_1^{\top 2}\widetilde{g}_{22}\\ e_2^{\top 1}\widetilde{g}_{11} + e_2^{\top 2}\widetilde{g}_{21} & e_2^{\top 1}\widetilde{g}_{12} + e_2^{\top 2}\widetilde{g}_{22}\end{bmatrix}\begin{bmatrix}e^1{}_1 & e^1{}_2\\ e^2{}_1 & e^2{}_2\end{bmatrix} \end{cases}$$

more correct ∵ index relay

$$\begin{bmatrix}e^1{}_1 & e^1{}_2\\ e^2{}_1 & e^2{}_2\end{bmatrix}^\top = \begin{bmatrix}e_1^{\top 1} & e_1^{\top 2}\\ e_2^{\top 1} & e_2^{\top 2}\end{bmatrix}$$

more correct ∵ index relay

$$e^{i}{}_{j}^{\top} = (e^i{}_j)^\top \leftrightarrow \begin{bmatrix}e^1{}_1 & e^1{}_2\\ e^2{}_1 & e^2{}_2\end{bmatrix}^\top = \begin{bmatrix}e_1^{\top 1} & e_1^{\top 2}\\ e_2^{\top 1} & e_2^{\top 2}\end{bmatrix} \leftrightarrow e_i^{\top j} = (e^\top)_i{}^j$$

more correct ∵ index relay

$$\vec{v} = \vec{e}_j v^j = \tilde{\vec{e}}_j \widetilde{v}^j$$

$$= \begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\begin{bmatrix}v^1\\ v^2\end{bmatrix} = \vec{e}_1 v^1 + \vec{e}_2 v^2 = \begin{bmatrix}\tilde{\vec{e}}_1 & \tilde{\vec{e}}_2\end{bmatrix}\begin{bmatrix}\widetilde{v}^1\\ \widetilde{v}^2\end{bmatrix} = \tilde{\vec{e}}_1\widetilde{v}^1 + \tilde{\vec{e}}_2\widetilde{v}^2$$

$$\begin{cases}\vec{e}_j = \tilde{\vec{e}}_i e^i{}_j\\ v^i = \widetilde{e}^i{}_j\widetilde{v}^j\end{cases} = \begin{bmatrix}\tilde{\vec{e}}_1 & \tilde{\vec{e}}_2\end{bmatrix}\begin{bmatrix}e^1{}_1 & e^1{}_2\\ e^2{}_1 & e^2{}_2\end{bmatrix}\begin{bmatrix}\widetilde{e}^1{}_1 & \widetilde{e}^1{}_2\\ \widetilde{e}^2{}_1 & \widetilde{e}^2{}_2\end{bmatrix}\begin{bmatrix}\widetilde{v}^1\\ \widetilde{v}^2\end{bmatrix} \qquad \begin{cases}\tilde{\vec{e}}_j = \vec{e}_i\widetilde{e}^i{}_j\\ \widetilde{v}^i = e^i{}_j v^j\end{cases} = \begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\begin{bmatrix}\widetilde{e}^1{}_1 & \widetilde{e}^1{}_2\\ \widetilde{e}^2{}_1 & \widetilde{e}^2{}_2\end{bmatrix}\begin{bmatrix}e^1{}_1 & e^1{}_2\\ e^2{}_1 & e^2{}_2\end{bmatrix}\begin{bmatrix}v^1\\ v^2\end{bmatrix}$$

$$= \begin{bmatrix}\tilde{\vec{e}}_1 & \tilde{\vec{e}}_2\end{bmatrix}\begin{bmatrix}\widetilde{v}^1\\ \widetilde{v}^2\end{bmatrix} = \begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\begin{bmatrix}v^1\\ v^2\end{bmatrix}$$

$$\downarrow$$

$$\overleftarrow{v} = \vec{v}\cdot\vec{w} = g_{ij}\overleftarrow{\epsilon}^i\overleftarrow{\epsilon}^j\vec{e}_k v^k\vec{e}_\ell w^\ell = \widetilde{g}_{ij}\tilde{\overleftarrow{\epsilon}}^i\tilde{\overleftarrow{\epsilon}}^j\tilde{\vec{e}}_k\widetilde{v}^k\tilde{\vec{e}}_\ell\widetilde{w}^\ell$$

$$= g_{ij}\overleftarrow{\epsilon}^i\delta^j{}_k v^k\vec{e}_\ell w^\ell = \widetilde{g}_{ij}\tilde{\overleftarrow{\epsilon}}^i\delta^j{}_k\widetilde{v}^k\tilde{\vec{e}}_\ell\widetilde{w}^\ell$$

$$= g_{ij}\overleftarrow{\epsilon}^i v^j\vec{e}_\ell w^\ell = \widetilde{g}_{ij}\tilde{\overleftarrow{\epsilon}}^i\widetilde{v}^j\tilde{\vec{e}}_\ell\widetilde{w}^\ell$$

$$= g_{ij}v^j\overleftarrow{\epsilon}^i\vec{e}_\ell w^\ell = \widetilde{g}_{ij}\widetilde{v}^j\tilde{\overleftarrow{\epsilon}}^i\tilde{\vec{e}}_\ell\widetilde{w}^\ell$$

$$\downarrow$$

$$\overleftarrow{v} = \vec{v}\cdot\square = g_{ij}v^j\overleftarrow{\epsilon}^i\square_\ell\square^\ell = \widetilde{g}_{ij}\widetilde{v}^j\tilde{\overleftarrow{\epsilon}}^i\widetilde{\square}_\ell\widetilde{\square}^\ell$$

$$= g_{ij}v^j\overleftarrow{\epsilon}^i = \widetilde{g}_{ij}\widetilde{v}^j\tilde{\overleftarrow{\epsilon}}^i$$

$$= \begin{bmatrix}v^1 & v^2\end{bmatrix}\begin{bmatrix}g_{11} & g_{12}\\ g_{21} & g_{22}\end{bmatrix}\begin{bmatrix}\overleftarrow{\epsilon}^1\\ \overleftarrow{\epsilon}^2\end{bmatrix} = \begin{bmatrix}v^1 & v^2\end{bmatrix}\begin{bmatrix}\vec{e}_1\cdot\vec{e}_1 & \vec{e}_1\cdot\vec{e}_2\\ \vec{e}_2\cdot\vec{e}_1 & \vec{e}_2\cdot\vec{e}_2\end{bmatrix}\begin{bmatrix}\overleftarrow{\epsilon}^1\\ \overleftarrow{\epsilon}^2\end{bmatrix} = \begin{bmatrix}v^1 & v^2\end{bmatrix}\begin{bmatrix}\vec{e}_1\cdot\\ \vec{e}_2\cdot\end{bmatrix}\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}\begin{bmatrix}\overleftarrow{\epsilon}^1\\ \overleftarrow{\epsilon}^2\end{bmatrix}$$

$$= \begin{bmatrix}\widetilde{v}^1 & \widetilde{v}^2\end{bmatrix}\begin{bmatrix}\widetilde{g}_{11} & \widetilde{g}_{12}\\ \widetilde{g}_{21} & \widetilde{g}_{22}\end{bmatrix}\begin{bmatrix}\tilde{\overleftarrow{\epsilon}}^1\\ \tilde{\overleftarrow{\epsilon}}^2\end{bmatrix} = \begin{bmatrix}\widetilde{v}^1 & \widetilde{v}^2\end{bmatrix}\begin{bmatrix}\tilde{\vec{e}}_1\cdot\tilde{\vec{e}}_1 & \tilde{\vec{e}}_1\cdot\tilde{\vec{e}}_2\\ \tilde{\vec{e}}_2\cdot\tilde{\vec{e}}_1 & \tilde{\vec{e}}_2\cdot\tilde{\vec{e}}_2\end{bmatrix}\begin{bmatrix}\tilde{\overleftarrow{\epsilon}}^1\\ \tilde{\overleftarrow{\epsilon}}^2\end{bmatrix} = \begin{bmatrix}\widetilde{v}^1 & \widetilde{v}^2\end{bmatrix}\begin{bmatrix}\tilde{\vec{e}}_1\cdot\\ \tilde{\vec{e}}_2\cdot\end{bmatrix}\begin{bmatrix}\tilde{\vec{e}}_1 & \tilde{\vec{e}}_2\end{bmatrix}\begin{bmatrix}\tilde{\overleftarrow{\epsilon}}^1\\ \tilde{\overleftarrow{\epsilon}}^2\end{bmatrix}$$

$$\leftrightarrow \begin{bmatrix}v^1 & v^2\end{bmatrix}\begin{bmatrix}g_{11} & g_{12}\\ g_{21} & g_{22}\end{bmatrix} = \begin{bmatrix}v^1 & v^2\end{bmatrix}\begin{bmatrix}\vec{e}_1\cdot\vec{e}_1 & \vec{e}_1\cdot\vec{e}_2\\ \vec{e}_2\cdot\vec{e}_1 & \vec{e}_2\cdot\vec{e}_2\end{bmatrix} = \begin{bmatrix}v^1 & v^2\end{bmatrix}\begin{bmatrix}\vec{e}_1\cdot\\ \vec{e}_2\cdot\end{bmatrix}\begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix}$$

$$= \begin{bmatrix}\widetilde{v}^1 & \widetilde{v}^2\end{bmatrix}\begin{bmatrix}\widetilde{g}_{11} & \widetilde{g}_{12}\\ \widetilde{g}_{21} & \widetilde{g}_{22}\end{bmatrix} = \begin{bmatrix}\widetilde{v}^1 & \widetilde{v}^2\end{bmatrix}\begin{bmatrix}\tilde{\vec{e}}_1\cdot\tilde{\vec{e}}_1 & \tilde{\vec{e}}_1\cdot\tilde{\vec{e}}_2\\ \tilde{\vec{e}}_2\cdot\tilde{\vec{e}}_1 & \tilde{\vec{e}}_2\cdot\tilde{\vec{e}}_2\end{bmatrix} = \begin{bmatrix}\widetilde{v}^1 & \widetilde{v}^2\end{bmatrix}\begin{bmatrix}\tilde{\vec{e}}_1\cdot\\ \tilde{\vec{e}}_2\cdot\end{bmatrix}\begin{bmatrix}\tilde{\vec{e}}_1 & \tilde{\vec{e}}_2\end{bmatrix}$$

$$g_{ij}v^j\overleftarrow{\epsilon}^i = v_i\overleftarrow{\epsilon}^i \leftrightarrow v_i = g_{ij}v^j$$

$$\widetilde{g}_{ij}\widetilde{v}^j\tilde{\overleftarrow{\epsilon}}^i = \widetilde{v}_i\tilde{\overleftarrow{\epsilon}}^i \leftrightarrow \widetilde{v}_i = \widetilde{g}_{ij}\widetilde{v}^j$$

$$v^i \not\equiv v_i = g_{ij} v^j$$

$$\widetilde{v}^i \not\equiv \widetilde{v}_i = \widetilde{g}_{ij} \widetilde{v}^j$$

$$\mathcal{V} \ni \vec{v} = \vec{e}_j v^j = \left|\vec{v}\right\rangle = \left|\vec{e}_j\right\rangle v^j = \left|\vec{e}_j v^j\right\rangle$$

$$\mathcal{V}^* \ni \overleftarrow{v} = \vec{v} \cdot \square = \left\langle\overleftarrow{v}\right| = v_i \left\langle\overleftarrow{\epsilon}^i\right| = \left\langle v_i \overleftarrow{\epsilon}^i\right| = \left\langle g_{ij} v^j \overleftarrow{\epsilon}^i\right| = g_{ij} v^j \left\langle\overleftarrow{\epsilon}^i\right|$$

$$\left\langle\vec{v}\right| = \begin{cases} \left\langle\vec{v}\right| = \left\langle\vec{e}_i v^i\right| = v^i \left\langle\vec{e}_i\right| = v^i \vec{e}_i \cdot = v^i \vec{e}_i \cdot \\ \left\langle\overleftarrow{v}\right| = \left\langle v_i \overleftarrow{\epsilon}^i\right| = v_i \left\langle\overleftarrow{\epsilon}^i\right| = v_i \overleftarrow{\epsilon}^i = g_{ij} v^j \overleftarrow{\epsilon}^i \end{cases}$$

$$\left\langle\vec{v}\right|\left|\vec{u}\right\rangle = \begin{cases} \left\langle\vec{v}|\vec{u}\right\rangle = \left\langle\vec{e}_i v^i|\vec{e}_j u^j\right\rangle = v^i \left\langle\vec{e}_i|\vec{e}_j\right\rangle u^j = v^i \vec{e}_i \cdot \vec{e}_j u^j = v^i \left(\vec{e}_i \cdot \vec{e}_j\right) u^j = v^i g_{ij} u^j \\ \left\langle\overleftarrow{v}|\vec{u}\right\rangle = \left\langle v_i \overleftarrow{\epsilon}^i|\vec{e}_j u^j\right\rangle = v_i \left\langle\overleftarrow{\epsilon}^i|\vec{e}_j\right\rangle u^j = v_i \overleftarrow{\epsilon}^i \vec{e}_j u^j = g_{ij} v^j \overleftarrow{\epsilon}^i \vec{e}_j u^j = g_{ij} v^j \delta^i{}_j u^j = g_{ij} v^i u^j = v^i g_{ij} u^j \end{cases}$$

$$\begin{aligned} \vec{v} \cdot \vec{v} &= \left(\vec{v}\cdot\right) \vec{v} = \left\langle\overleftarrow{v}|\vec{v}\right\rangle \\ &= \left\langle\overleftarrow{v}\right|\left|\vec{v}\right\rangle = \left\langle v_i \overleftarrow{\epsilon}^i\right|\left|\vec{e}_j v^j\right\rangle = v_i \left\langle\overleftarrow{\epsilon}^i\right|\left|\vec{e}_j\right\rangle v^j = g_{ik} v^k \left\langle\overleftarrow{\epsilon}^i\right|\left|\vec{e}_j\right\rangle v^j = g_{ik} v^k \delta^i{}_j v^j \\ &= g_{jk} v^k v^j \\ &= g_{jk} v^j v^k && v^k v^j = v^j v^k \\ &= g_{ij} v^i v^j && j \to i,\quad k \to j \end{aligned}$$

$$\mathcal{V}^* \ni \overleftarrow{u} = \vec{u} \cdot \square = \left\langle\overleftarrow{u}\right| = u_i \left\langle\overleftarrow{\epsilon}^i\right| = \left\langle u_i \overleftarrow{\epsilon}^i\right| = \left\langle g_{ij} u^j \overleftarrow{\epsilon}^i\right| = g_{ij} u^j \left\langle\overleftarrow{\epsilon}^i\right|$$

$$\begin{aligned} \vec{u} \cdot \vec{v} &= \left(\vec{u}\cdot\right) \vec{v} = \left\langle\overleftarrow{u}|\vec{v}\right\rangle \\ &= \left\langle\overleftarrow{u}\right|\left|\vec{v}\right\rangle = \left\langle u_i \overleftarrow{\epsilon}^i\right|\left|\vec{e}_j v^j\right\rangle = u_i \left\langle\overleftarrow{\epsilon}^i\right|\left|\vec{e}_j\right\rangle v^j = g_{ik} u^k \left\langle\overleftarrow{\epsilon}^i\right|\left|\vec{e}_j\right\rangle v^j = g_{ik} u^k \delta^i{}_j v^j \\ &= g_{jk} u^k v^j \\ &= g_{kj} u^k v^j && g_{jk} = g_{kj} \\ &= g_{ij} u^i v^j && k \to i \end{aligned}$$

$$g = [g_{ij}] = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} \vec{e}_1 \cdot \vec{e}_1 & \vec{e}_1 \cdot \vec{e}_2 \\ \vec{e}_2 \cdot \vec{e}_1 & \vec{e}_2 \cdot \vec{e}_2 \end{bmatrix} = \begin{bmatrix} \vec{e}_1 \cdot \\ \vec{e}_2 \cdot \end{bmatrix} \begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix}$$

$$\widetilde{g} = [\widetilde{g}_{ij}] = \begin{bmatrix} \widetilde{g}_{11} & \widetilde{g}_{12} \\ \widetilde{g}_{21} & \widetilde{g}_{22} \end{bmatrix} = \begin{bmatrix} \widetilde{\vec{e}}_1 \cdot \widetilde{\vec{e}}_1 & \widetilde{\vec{e}}_1 \cdot \widetilde{\vec{e}}_2 \\ \widetilde{\vec{e}}_2 \cdot \widetilde{\vec{e}}_1 & \widetilde{\vec{e}}_2 \cdot \widetilde{\vec{e}}_2 \end{bmatrix} = \begin{bmatrix} \widetilde{\vec{e}}_1 \cdot \\ \widetilde{\vec{e}}_2 \cdot \end{bmatrix} \begin{bmatrix} \widetilde{\vec{e}}_1 & \widetilde{\vec{e}}_2 \end{bmatrix}$$

$$\overline{g} = g^{-1} = [g_{ij}]^{-1} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}^{-1} = \begin{bmatrix} \vec{e}_1 \cdot \vec{e}_1 & \vec{e}_1 \cdot \vec{e}_2 \\ \vec{e}_2 \cdot \vec{e}_1 & \vec{e}_2 \cdot \vec{e}_2 \end{bmatrix}^{-1} = \left(\begin{bmatrix} \vec{e}_1 \cdot \\ \vec{e}_2 \cdot \end{bmatrix} \begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix}\right)^{-1}$$

$$\overline{\widetilde{g}} = \widetilde{g}^{-1} = [\widetilde{g}_{ij}]^{-1} = \begin{bmatrix} \widetilde{g}_{11} & \widetilde{g}_{12} \\ \widetilde{g}_{21} & \widetilde{g}_{22} \end{bmatrix}^{-1} = \begin{bmatrix} \widetilde{\vec{e}}_1 \cdot \widetilde{\vec{e}}_1 & \widetilde{\vec{e}}_1 \cdot \widetilde{\vec{e}}_2 \\ \widetilde{\vec{e}}_2 \cdot \widetilde{\vec{e}}_1 & \widetilde{\vec{e}}_2 \cdot \widetilde{\vec{e}}_2 \end{bmatrix}^{-1} = \left(\begin{bmatrix} \widetilde{\vec{e}}_1 \cdot \\ \widetilde{\vec{e}}_2 \cdot \end{bmatrix} \begin{bmatrix} \widetilde{\vec{e}}_1 & \widetilde{\vec{e}}_2 \end{bmatrix}\right)^{-1}$$

$$\mathfrak{g} = \begin{bmatrix} \mathfrak{g}^{11} & \mathfrak{g}^{12} \\ \mathfrak{g}^{21} & \mathfrak{g}^{22} \end{bmatrix} = \overline{g} = g^{-1} = [g_{ij}]^{-1} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}^{-1} = \begin{bmatrix} \vec{e}_1 \cdot \vec{e}_1 & \vec{e}_1 \cdot \vec{e}_2 \\ \vec{e}_2 \cdot \vec{e}_1 & \vec{e}_2 \cdot \vec{e}_2 \end{bmatrix}^{-1} = \left(\begin{bmatrix} \vec{e}_1 \cdot \\ \vec{e}_2 \cdot \end{bmatrix} \begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix}\right)^{-1}$$

$$\widetilde{\mathfrak{g}} = \begin{bmatrix} \widetilde{\mathfrak{g}}^{11} & \widetilde{\mathfrak{g}}^{12} \\ \widetilde{\mathfrak{g}}^{21} & \widetilde{\mathfrak{g}}^{22} \end{bmatrix} = \overline{\widetilde{g}} = \widetilde{g}^{-1} = [\widetilde{g}_{ij}]^{-1} = \begin{bmatrix} \widetilde{g}_{11} & \widetilde{g}_{12} \\ \widetilde{g}_{21} & \widetilde{g}_{22} \end{bmatrix}^{-1} = \begin{bmatrix} \widetilde{\vec{e}}_1 \cdot \widetilde{\vec{e}}_1 & \widetilde{\vec{e}}_1 \cdot \widetilde{\vec{e}}_2 \\ \widetilde{\vec{e}}_2 \cdot \widetilde{\vec{e}}_1 & \widetilde{\vec{e}}_2 \cdot \widetilde{\vec{e}}_2 \end{bmatrix}^{-1} = \left(\begin{bmatrix} \widetilde{\vec{e}}_1 \cdot \\ \widetilde{\vec{e}}_2 \cdot \end{bmatrix} \begin{bmatrix} \widetilde{\vec{e}}_1 & \widetilde{\vec{e}}_2 \end{bmatrix}\right)^{-1}$$

$$v_j = v^i g_{ij} \leftrightarrow \begin{bmatrix} v^1 & v^2 \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}$$

$$v_j \mathfrak{g}^{jk} = v^i g_{ij} \mathfrak{g}^{jk} \leftrightarrow \begin{bmatrix} v^1 & v^2 \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} \mathfrak{g}^{11} & \mathfrak{g}^{12} \\ \mathfrak{g}^{21} & \mathfrak{g}^{22} \end{bmatrix} = \begin{bmatrix} v^1 & v^2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} v^1 & v^2 \end{bmatrix}$$

$$v_j \mathfrak{g}^{jk} = v^i g_{ij} \mathfrak{g}^{jk} = v^k \leftrightarrow \begin{bmatrix} v^1 & v^2 \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} \mathfrak{g}^{11} & \mathfrak{g}^{12} \\ \mathfrak{g}^{21} & \mathfrak{g}^{22} \end{bmatrix} = \begin{bmatrix} v^1 & v^2 \end{bmatrix}$$

$$\Downarrow$$

$$g_{ij}\mathfrak{g}^{jk} = \delta_i{}^k \overset{\substack{i \to j \\ j \to k \\ k \to i}}{\Rightarrow} g_{jk}\mathfrak{g}^{ki} = \delta_j{}^i \overset{\substack{g_{jk} = g_{kj} \\ \mathfrak{g}^{ki} = \mathfrak{g}^{ik}}}{\Rightarrow} \mathfrak{g}^{ik} g_{kj} = \delta^i{}_j$$

$$v_j = v^i g_{ij} \qquad \widetilde{v}_j = \widetilde{v}^i \widetilde{g}_{ij}$$

$$v_j \mathfrak{g}^{jk} = v^k \qquad \widetilde{v}_j \widetilde{\mathfrak{g}}^{jk} = \widetilde{v}^k \overset{k \to i}{\Rightarrow} v_j \mathfrak{g}^{ji} = v^i \qquad \widetilde{v}_j \widetilde{\mathfrak{g}}^{ji} = \widetilde{v}^i$$

$$g_{jk}\mathfrak{g}^{ki} = \delta_j{}^i = \widetilde{g}_{jk}\widetilde{\mathfrak{g}}^{ki}$$

$$v_i = g_{ij} v^j \leftrightarrow \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix}$$

$$\mathfrak{g}^{ki} v_i = \mathfrak{g}^{ki} g_{ij} v^j \leftrightarrow \begin{bmatrix} \mathfrak{g}^{11} & \mathfrak{g}^{12} \\ \mathfrak{g}^{21} & \mathfrak{g}^{22} \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \begin{bmatrix} v^1 \\ v^2 \end{bmatrix}$$

$$\mathfrak{g}^{ki} v_i = \mathfrak{g}^{ki} g_{ij} v^j = v^k \leftrightarrow \begin{bmatrix} \mathfrak{g}^{11} & \mathfrak{g}^{12} \\ \mathfrak{g}^{21} & \mathfrak{g}^{22} \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \begin{bmatrix} v^1 \\ v^2 \end{bmatrix}$$

$$\Downarrow$$

$$\mathfrak{g}^{ki} g_{ij} = \delta^k{}_j \overset{\substack{i \to k \\ k \to i}}{\Rightarrow} \mathfrak{g}^{ik} g_{kj} = \delta^i{}_j \overset{\substack{g_{jk} = g_{kj} \\ \mathfrak{g}^{ki} = \mathfrak{g}^{ik}}}{\Rightarrow} g_{jk}\mathfrak{g}^{ki} = \delta_j{}^i$$

$$v_i = g_{ij} v^j \qquad \widetilde{v}_i = \widetilde{g}_{ij} \widetilde{v}^j$$

$$\mathfrak{g}^{ki} v_i = v^k \qquad \widetilde{\mathfrak{g}}^{ki} \widetilde{v}_i = \widetilde{v}^k \overset{k \to i}{\Rightarrow} \mathfrak{g}^{ij} v_j = v^i \qquad \widetilde{\mathfrak{g}}^{ij} \widetilde{v}_j = \widetilde{v}^i$$

$$\mathfrak{g}^{ik} g_{kj} = \delta^i{}_j = \widetilde{\mathfrak{g}}^{ik} \widetilde{g}_{kj}$$

matrix representation example summary for lowering (flat) and raising (sharp) index(ices) with metric and inverse metric tensor

$$v_j = v^i g_{ij} \qquad \widetilde{v}_j = \widetilde{v}^i \widetilde{g}_{ij}$$

$$\begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} v^1 & v^2 \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \qquad \begin{bmatrix} \widetilde{v}_1 & \widetilde{v}_2 \end{bmatrix} = \begin{bmatrix} \widetilde{v}^1 & \widetilde{v}^2 \end{bmatrix} \begin{bmatrix} \widetilde{g}_{11} & \widetilde{g}_{12} \\ \widetilde{g}_{21} & \widetilde{g}_{22} \end{bmatrix}$$

$$v_j \mathfrak{g}^{ji} = v^i \qquad \widetilde{v}_j \widetilde{\mathfrak{g}}^{ji} = \widetilde{v}^i$$

$$\begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} \mathfrak{g}^{11} & \mathfrak{g}^{12} \\ \mathfrak{g}^{21} & \mathfrak{g}^{22} \end{bmatrix} = \begin{bmatrix} v^1 & v^2 \end{bmatrix} \qquad \begin{bmatrix} \widetilde{v}_1 & \widetilde{v}_2 \end{bmatrix} \begin{bmatrix} \widetilde{\mathfrak{g}}^{11} & \widetilde{\mathfrak{g}}^{12} \\ \widetilde{\mathfrak{g}}^{21} & \widetilde{\mathfrak{g}}^{22} \end{bmatrix} = \begin{bmatrix} \widetilde{v}^1 & \widetilde{v}^2 \end{bmatrix}$$

$$g_{jk}\mathfrak{g}^{ki} = \delta_j{}^i = \widetilde{g}_{jk}\widetilde{\mathfrak{g}}^{ki}$$

$$\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} \mathfrak{g}^{11} & \mathfrak{g}^{12} \\ \mathfrak{g}^{21} & \mathfrak{g}^{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \widetilde{g}_{11} & \widetilde{g}_{12} \\ \widetilde{g}_{21} & \widetilde{g}_{22} \end{bmatrix} \begin{bmatrix} \widetilde{\mathfrak{g}}^{11} & \widetilde{\mathfrak{g}}^{12} \\ \widetilde{\mathfrak{g}}^{21} & \widetilde{\mathfrak{g}}^{22} \end{bmatrix}$$

$$v_i = g_{ij} v^j \qquad \widetilde{v}_i = \widetilde{g}_{ij} \widetilde{v}^j$$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \qquad \begin{bmatrix} \widetilde{v}_1 \\ \widetilde{v}_2 \end{bmatrix} = \begin{bmatrix} \widetilde{g}_{11} & \widetilde{g}_{12} \\ \widetilde{g}_{21} & \widetilde{g}_{22} \end{bmatrix} \begin{bmatrix} \widetilde{v}^1 \\ \widetilde{v}^2 \end{bmatrix}$$

$$\mathfrak{g}^{ij} v_j = v^i \qquad \widetilde{\mathfrak{g}}^{ij} \widetilde{v}_j = \widetilde{v}^i$$

$$\begin{bmatrix} \mathfrak{g}^{11} & \mathfrak{g}^{12} \\ \mathfrak{g}^{21} & \mathfrak{g}^{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \qquad \begin{bmatrix} \widetilde{\mathfrak{g}}^{11} & \widetilde{\mathfrak{g}}^{12} \\ \widetilde{\mathfrak{g}}^{21} & \widetilde{\mathfrak{g}}^{22} \end{bmatrix} \begin{bmatrix} \widetilde{v}_1 \\ \widetilde{v}_2 \end{bmatrix} = \begin{bmatrix} \widetilde{v}^1 \\ \widetilde{v}^2 \end{bmatrix}$$

$$\mathfrak{g}^{ik} g_{kj} = \delta^i{}_j = \widetilde{\mathfrak{g}}^{ik} \widetilde{g}_{kj}$$

$$\begin{bmatrix} \mathfrak{g}^{11} & \mathfrak{g}^{12} \\ \mathfrak{g}^{21} & \mathfrak{g}^{22} \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \widetilde{\mathfrak{g}}^{11} & \widetilde{\mathfrak{g}}^{12} \\ \widetilde{\mathfrak{g}}^{21} & \widetilde{\mathfrak{g}}^{22} \end{bmatrix} \begin{bmatrix} \widetilde{g}_{11} & \widetilde{g}_{12} \\ \widetilde{g}_{21} & \widetilde{g}_{22} \end{bmatrix}$$

lowering (flat) and raising (sharp) index(ices)

$$\overleftarrow{v} = \vec{v}\cdot\square = g\left(\vec{v},\square\right) = \left\langle\vec{v}\right| = \flat\vec{v} = \flat\vec{e}_i v^i = v_i\overleftarrow{\epsilon}^i = g_{ij}v^j\overleftarrow{\epsilon}^i$$

$$\vec{\alpha} = \square\cdot\overleftarrow{\alpha} = \mathfrak{g}\left(\overleftarrow{\alpha},\square\right) = \left|\overleftarrow{\alpha}\right\rangle = \sharp\overleftarrow{\alpha} = \sharp\alpha_i\overleftarrow{\epsilon}^i = \vec{e}_i\alpha^i = \vec{e}_i\alpha_j\mathfrak{g}^{ji}$$

matrix representation example summary for lowering (flat) and raising (sharp) index(ices) with metric and inverse metric tensor in vector space and dual space

$$\overleftarrow{v} = \vec{v}\cdot\square = g\left(\vec{v},\square\right) = \left\langle\vec{v}\right| = \flat\vec{v} = \flat\vec{e}_i v^i = v_i\overleftarrow{\epsilon}^i = v^j g_{ji}\overleftarrow{\epsilon}^i = g_{ij}v^j\overleftarrow{\epsilon}^i$$

$$\begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} v^1 & v^2 \end{bmatrix}\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}$$

$$\begin{bmatrix} v_1 & v_2 \end{bmatrix}\begin{bmatrix} \overleftarrow{\epsilon}^1 \\ \overleftarrow{\epsilon}^2 \end{bmatrix} = \begin{bmatrix} v^1 & v^2 \end{bmatrix}\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}\begin{bmatrix} \overleftarrow{\epsilon}^1 \\ \overleftarrow{\epsilon}^2 \end{bmatrix} = \begin{bmatrix} v^1 & v^2 \end{bmatrix}\begin{bmatrix} \vec{e}_1\cdot \\ \vec{e}_2\cdot \end{bmatrix}\begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix}\begin{bmatrix} \overleftarrow{\epsilon}^1 \\ \overleftarrow{\epsilon}^2 \end{bmatrix}$$

$$\vec{\alpha} = \square\cdot\overleftarrow{\alpha} = \mathfrak{g}\left(\overleftarrow{\alpha},\square\right) = \left|\vec{\alpha}\right\rangle = \sharp\overleftarrow{\alpha} = \sharp\alpha_i\overleftarrow{\epsilon}^i = \vec{e}_i\alpha^i = \vec{e}_i\mathfrak{g}^{ij}\alpha_j = \vec{e}_i\alpha_j\mathfrak{g}^{ji}$$

$$\begin{bmatrix} \mathfrak{g}^{11} & \mathfrak{g}^{12} \\ \mathfrak{g}^{21} & \mathfrak{g}^{22} \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \alpha^1 \\ \alpha^2 \end{bmatrix}$$

$$\begin{bmatrix} \alpha^1 \\ \alpha^2 \end{bmatrix} = \begin{bmatrix} \mathfrak{g}^{11} & \mathfrak{g}^{12} \\ \mathfrak{g}^{21} & \mathfrak{g}^{22} \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$

$$\begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix}\begin{bmatrix} \alpha^1 \\ \alpha^2 \end{bmatrix} = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix}\begin{bmatrix} \mathfrak{g}^{11} & \mathfrak{g}^{12} \\ \mathfrak{g}^{21} & \mathfrak{g}^{22} \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix}\begin{bmatrix} \overleftarrow{\epsilon}^1 \\ \overleftarrow{\epsilon}^2 \end{bmatrix}\begin{bmatrix} \cdot\overleftarrow{\epsilon}^1 & \cdot\overleftarrow{\epsilon}^2 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$

$$\begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix}\begin{bmatrix} L^1{}_1 & L^1{}_2 \\ L^2{}_1 & L^2{}_2 \end{bmatrix}\begin{bmatrix} \overleftarrow{\epsilon}^1 \\ \overleftarrow{\epsilon}^2 \end{bmatrix}$$

$$= \begin{bmatrix} \widetilde{\vec{e}}_1 & \widetilde{\vec{e}}_2 \end{bmatrix}\begin{bmatrix} e^1{}_1 & e^1{}_2 \\ e^2{}_1 & e^2{}_2 \end{bmatrix}\begin{bmatrix} L^1{}_1 & L^1{}_2 \\ L^2{}_1 & L^2{}_2 \end{bmatrix}\begin{bmatrix} \widetilde{e}^1{}_1 & \widetilde{e}^1{}_2 \\ \widetilde{e}^2{}_1 & \widetilde{e}^2{}_2 \end{bmatrix}\begin{bmatrix} \widetilde{\overleftarrow{\epsilon}}^1 \\ \widetilde{\overleftarrow{\epsilon}}^2 \end{bmatrix}$$

$$= \begin{bmatrix} \widetilde{\vec{e}}_1 & \widetilde{\vec{e}}_2 \end{bmatrix}\begin{bmatrix} \widetilde{L}^1{}_1 & \widetilde{L}^1{}_2 \\ \widetilde{L}^2{}_1 & \widetilde{L}^2{}_2 \end{bmatrix}\begin{bmatrix} \widetilde{\overleftarrow{\epsilon}}^1 \\ \widetilde{\overleftarrow{\epsilon}}^2 \end{bmatrix} \Rightarrow \begin{bmatrix} L^1{}_1 & L^1{}_2 \\ L^2{}_1 & L^2{}_2 \end{bmatrix} = \begin{bmatrix} \widetilde{e}^1{}_1 & \widetilde{e}^1{}_2 \\ \widetilde{e}^2{}_1 & \widetilde{e}^2{}_2 \end{bmatrix}\begin{bmatrix} \widetilde{L}^1{}_1 & \widetilde{L}^1{}_2 \\ \widetilde{L}^2{}_1 & \widetilde{L}^2{}_2 \end{bmatrix}\begin{bmatrix} e^1{}_1 & e^1{}_2 \\ e^2{}_1 & e^2{}_2 \end{bmatrix}$$

$$\begin{bmatrix} L^1{}_1 & L^1{}_2 \\ L^2{}_1 & L^2{}_2 \end{bmatrix} = \begin{bmatrix} \widetilde{e}^1{}_1 & \widetilde{e}^1{}_2 \\ \widetilde{e}^2{}_1 & \widetilde{e}^2{}_2 \end{bmatrix}\begin{bmatrix} \widetilde{L}^1{}_1 & \widetilde{L}^1{}_2 \\ \widetilde{L}^2{}_1 & \widetilde{L}^2{}_2 \end{bmatrix}\begin{bmatrix} e^1{}_1 & e^1{}_2 \\ e^2{}_1 & e^2{}_2 \end{bmatrix}$$

$$\begin{bmatrix} L^1{}_1 & L^1{}_2 \\ L^2{}_1 & L^2{}_2 \end{bmatrix}^\intercal = \left(\begin{bmatrix} \widetilde{e}^1{}_1 & \widetilde{e}^1{}_2 \\ \widetilde{e}^2{}_1 & \widetilde{e}^2{}_2 \end{bmatrix}\begin{bmatrix} \widetilde{L}^1{}_1 & \widetilde{L}^1{}_2 \\ \widetilde{L}^2{}_1 & \widetilde{L}^2{}_2 \end{bmatrix}\begin{bmatrix} e^1{}_1 & e^1{}_2 \\ e^2{}_1 & e^2{}_2 \end{bmatrix}\right)^\intercal$$

$$= \begin{bmatrix} e^1{}_1 & e^1{}_2 \\ e^2{}_1 & e^2{}_2 \end{bmatrix}^\intercal\begin{bmatrix} \widetilde{L}^1{}_1 & \widetilde{L}^1{}_2 \\ \widetilde{L}^2{}_1 & \widetilde{L}^2{}_2 \end{bmatrix}^\intercal\begin{bmatrix} \widetilde{e}^1{}_1 & \widetilde{e}^1{}_2 \\ \widetilde{e}^2{}_1 & \widetilde{e}^2{}_2 \end{bmatrix}^\intercal \qquad (ABC)^\intercal = C^\intercal B^\intercal A^\intercal$$

$$\begin{bmatrix} L^1{}_1 & L^1{}_2 \\ L^2{}_1 & L^2{}_2 \end{bmatrix}^\intercal = \begin{bmatrix} e^1{}_1 & e^1{}_2 \\ e^2{}_1 & e^2{}_2 \end{bmatrix}^\intercal\begin{bmatrix} \widetilde{L}^1{}_1 & \widetilde{L}^1{}_2 \\ \widetilde{L}^2{}_1 & \widetilde{L}^2{}_2 \end{bmatrix}^\intercal\begin{bmatrix} \widetilde{e}^1{}_1 & \widetilde{e}^1{}_2 \\ \widetilde{e}^2{}_1 & \widetilde{e}^2{}_2 \end{bmatrix}^\intercal$$

$L^i{}_j{}^\intercal = (L^i{}_j)^\intercal = L^j{}_i = L^{\intercal j}_i = (L^\intercal)_i{}^j$

$$e^i{}_j{}^\intercal = (e^i{}_j)^\intercal \leftrightarrow \begin{bmatrix} e^1{}_1 & e^1{}_2 \\ e^2{}_1 & e^2{}_2 \end{bmatrix}^\intercal = \begin{bmatrix} e^{\intercal 1}_1 & e^{\intercal 2}_1 \\ e^{\intercal 1}_2 & e^{\intercal 2}_2 \end{bmatrix} \leftrightarrow e^{\intercal j}_i = (e^\intercal)_i{}^j \qquad \text{more correct} \because \text{index relay}$$

$$L^i{}_j{}^\intercal = (L^i{}_j)^\intercal \leftrightarrow \begin{bmatrix} L^1{}_1 & L^1{}_2 \\ L^2{}_1 & L^2{}_2 \end{bmatrix}^\intercal = \begin{bmatrix} L^{\intercal 1}_1 & L^{\intercal 2}_1 \\ L^{\intercal 1}_2 & L^{\intercal 2}_2 \end{bmatrix} \leftrightarrow L^{\intercal j}_i = (L^\intercal)_i{}^j \qquad \text{more correct} \because \text{index relay}$$

$$\widetilde{L}^i{}_j{}^\intercal = \left(\widetilde{L}^i{}_j\right)^\intercal \leftrightarrow \begin{bmatrix} \widetilde{L}^1{}_1 & \widetilde{L}^1{}_2 \\ \widetilde{L}^2{}_1 & \widetilde{L}^2{}_2 \end{bmatrix}^\intercal = \begin{bmatrix} \widetilde{L}^{\intercal 1}_1 & \widetilde{L}^{\intercal 2}_1 \\ \widetilde{L}^{\intercal 1}_2 & \widetilde{L}^{\intercal 2}_2 \end{bmatrix} \leftrightarrow \widetilde{L}^{\intercal j}_i = \left(\widetilde{L}^\intercal\right)_i{}^j \qquad \text{more correct} \because \text{index relay}$$

$$\widetilde{e}^i{}_j{}^\intercal = (\widetilde{e}^i{}_j)^\intercal \leftrightarrow \begin{bmatrix} \widetilde{e}^1{}_1 & \widetilde{e}^1{}_2 \\ \widetilde{e}^2{}_1 & \widetilde{e}^2{}_2 \end{bmatrix}^\intercal = \begin{bmatrix} \widetilde{e}^{\intercal 1}_1 & \widetilde{e}^{\intercal 2}_1 \\ \widetilde{e}^{\intercal 1}_2 & \widetilde{e}^{\intercal 2}_2 \end{bmatrix} \leftrightarrow \widetilde{e}^{\intercal j}_i = (\widetilde{e}^\intercal)_i{}^j \qquad \text{more correct} \because \text{index relay}$$

$$L^j{}_i \leftrightarrow \begin{bmatrix} L^1{}_1 & L^2{}_1 \\ L^1{}_2 & L^2{}_2 \end{bmatrix} = \begin{bmatrix} L^1{}_1 & L^1{}_2 \\ L^2{}_1 & L^2{}_2 \end{bmatrix}^\intercal = \begin{bmatrix} L^{\intercal 1}_1 & L^{\intercal 2}_1 \\ L^{\intercal 1}_2 & L^{\intercal 2}_2 \end{bmatrix} \leftrightarrow L^{\intercal j}_i$$

$$L^{i\intercal}{}_{j} = (L^i{}_j)^\intercal = L^j{}_i = L^{\intercal j}_i = (L^\intercal)_i{}^j$$

$$\begin{bmatrix} L^1{}_1 & L^1{}_2 \\ L^2{}_1 & L^2{}_2 \end{bmatrix}^\intercal = \begin{bmatrix} e^1{}_1 & e^1{}_2 \\ e^2{}_1 & e^2{}_2 \end{bmatrix}^\intercal \begin{bmatrix} \widetilde{L}^1{}_1 & \widetilde{L}^1{}_2 \\ \widetilde{L}^2{}_1 & \widetilde{L}^2{}_2 \end{bmatrix}^\intercal \begin{bmatrix} \widetilde{e}^1{}_1 & \widetilde{e}^1{}_2 \\ \widetilde{e}^2{}_1 & \widetilde{e}^2{}_2 \end{bmatrix}^\intercal$$

$$\begin{aligned} L^{\intercal j}_i \leftrightarrow \begin{bmatrix} L^{\intercal 1}_1 & L^{\intercal 2}_1 \\ L^{\intercal 1}_2 & L^{\intercal 2}_2 \end{bmatrix} &= \begin{bmatrix} e^{\intercal 1}_1 & e^{\intercal 2}_1 \\ e^{\intercal 1}_2 & e^{\intercal 2}_2 \end{bmatrix} \begin{bmatrix} \widetilde{L}^{\intercal 1}_1 & \widetilde{L}^{\intercal 2}_1 \\ \widetilde{L}^{\intercal 1}_2 & \widetilde{L}^{\intercal 2}_2 \end{bmatrix} \begin{bmatrix} \widetilde{e}^{\intercal 1}_1 & \widetilde{e}^{\intercal 2}_1 \\ \widetilde{e}^{\intercal 1}_2 & \widetilde{e}^{\intercal 2}_2 \end{bmatrix} \leftrightarrow e^{\intercal h}_i \widetilde{L}^{\intercal k}_h \widetilde{e}^{\intercal j}_k \\ &= \begin{bmatrix} e^{\intercal 1}_1 \widetilde{L}^{\intercal 1}_1 + e^{\intercal 2}_1 \widetilde{L}^{\intercal 1}_2 & e^{\intercal 1}_1 \widetilde{L}^{\intercal 2}_1 + e^{\intercal 2}_1 \widetilde{L}^{\intercal 2}_2 \\ e^{\intercal 1}_2 \widetilde{L}^{\intercal 1}_1 + e^{\intercal 2}_2 \widetilde{L}^{\intercal 1}_2 & e^{\intercal 1}_2 \widetilde{L}^{\intercal 2}_1 + e^{\intercal 2}_2 \widetilde{L}^{\intercal 2}_2 \end{bmatrix} \begin{bmatrix} \widetilde{e}^{\intercal 1}_1 & \widetilde{e}^{\intercal 2}_1 \\ \widetilde{e}^{\intercal 1}_2 & \widetilde{e}^{\intercal 2}_2 \end{bmatrix} \\ &= \begin{bmatrix} e^{\intercal 1}_1 & e^{\intercal 2}_1 \\ e^{\intercal 1}_2 & e^{\intercal 2}_2 \end{bmatrix} \begin{bmatrix} \widetilde{L}^{\intercal 1}_1 \widetilde{e}^{\intercal 1}_1 + \widetilde{L}^{\intercal 2}_1 \widetilde{e}^{\intercal 1}_2 & \widetilde{L}^{\intercal 1}_1 \widetilde{e}^{\intercal 2}_1 + \widetilde{L}^{\intercal 2}_1 \widetilde{e}^{\intercal 2}_2 \\ \widetilde{L}^{\intercal 1}_2 \widetilde{e}^{\intercal 1}_1 + \widetilde{L}^{\intercal 2}_2 \widetilde{e}^{\intercal 1}_2 & \widetilde{L}^{\intercal 1}_2 \widetilde{e}^{\intercal 2}_1 + \widetilde{L}^{\intercal 2}_2 \widetilde{e}^{\intercal 2}_2 \end{bmatrix} \end{aligned}$$

$$L^{\intercal j}_i = e^{\intercal h}_i \widetilde{L}^{\intercal k}_h \widetilde{e}^{\intercal j}_k$$

$$\begin{aligned} L^{\intercal j}_i &= e^{\intercal h}_i \widetilde{L}^{\intercal k}_h \widetilde{e}^{\intercal j}_k \\ &\Downarrow \\ \widetilde{L}^{\intercal j}_i &= \widetilde{e}^{\intercal h}_i L^{\intercal k}_h e^{\intercal j}_k \end{aligned}$$

$$\widetilde{L}^{\intercal j}_i = \widetilde{e}^{\intercal h}_i L^{\intercal k}_h e^{\intercal j}_k$$

$$L^{\intercal i}{}_j = (L^\intercal)^i{}_j = \mathfrak{g}^{ih} L^{\intercal k}_h g_{kj} = g_{jk} L^k{}_h \mathfrak{g}^{hi} = L_j{}^i$$

$$\begin{aligned} L^{\intercal i}{}_j = (L^\intercal)^i{}_j =& \mathfrak{g}^{ih} L^{\intercal k}_h g_{kj} \\ =& \mathfrak{g}^{ih} (L^\intercal)_h{}^k g_{kj} \\ =& \mathfrak{g}^{ih} L^k{}_h g_{kj} \\ =& g_{kj} L^k{}_h \mathfrak{g}^{ih} \\ =& g_{jk} L^k{}_h \mathfrak{g}^{hi} \\ =& L_j{}^i \end{aligned}$$

g_{jk} lowering index(ies)
$\mathfrak{g}^{hi}$ raising index(ies)
$L^{\intercal k}_h = (L^\intercal)_h{}^k \Leftarrow L^{\intercal j}_i = (L^\intercal)_i{}^j$
$L^k{}_h = (L^\intercal)_h{}^k \Leftarrow L^j{}_i = L^{\intercal j}_i = (L^\intercal)_i{}^j$

$g_{jk} = g_{kj}$
$\mathfrak{g}^{hi} = \mathfrak{g}^{ih}$
g_{jk} lowering index(ies)
$\mathfrak{g}^{hi}$ raising index(ies)

$(0,1)$-tensor
covariant
basis vector

$$\begin{cases}\tilde{\vec{e}}_j = \vec{e}_i {\tilde{e}^i}_j \\ \vec{e}_j = \tilde{\vec{e}}_i {e^i}_j\end{cases}$$

covector component scalar

$$\begin{cases}\tilde{\alpha}_j = \alpha_i {\tilde{e}^i}_j \\ \alpha_j = \tilde{\alpha}_i {e^i}_j\end{cases}$$

$(0,2)$-tensor
bilinear form = 2-form: including metric tensor

$$\begin{cases}\tilde{b}_{ij} = b_{hk} {\tilde{e}^h}_i {\tilde{e}^k}_j \\ b_{ij} = \tilde{b}_{hk} {e^h}_i {e^k}_j\end{cases}$$

$(1,0)$-tensor
contravariant
vector component scalar

$$\begin{cases}\tilde{v}^i = {e^i}_j v^j \\ v^i = {\tilde{e}^i}_j \tilde{v}^j\end{cases}$$

dual basis covector

$$\begin{cases}\tilde{\overleftarrow{\epsilon}}^i = {e^i}_j \overleftarrow{\epsilon}^j \\ \overleftarrow{\epsilon}^i = {\tilde{e}^i}_j \tilde{\overleftarrow{\epsilon}}^j\end{cases}$$

$(2,0)$-tensor
inverse metric tensor

$$\begin{cases}\tilde{\mathfrak{g}}^{ij} = {e^i}_h {e^j}_k \mathfrak{g}^{hk} \\ \mathfrak{g}^{ij} = {\tilde{e}^i}_h {\tilde{e}^j}_k \tilde{\mathfrak{g}}^{hk}\end{cases}$$

$(1,1)$-tensor

linear map

$$\begin{cases}{\tilde{L}^i}_j = {e^i}_h {L^h}_k {\tilde{e}^k}_j \\ {L^i}_j = {\tilde{e}^i}_h {\tilde{L}^h}_k {e^k}_j\end{cases}$$

transpose linear map

$$\begin{cases}\tilde{L}_i^{\intercal j} = \tilde{e}_i^{\intercal h} L_h^{\intercal k} e_k^{\intercal j} \\ L_i^{\intercal j} = e_i^{\intercal h} \tilde{L}_h^{\intercal k} \tilde{e}_k^{\intercal j}\end{cases}$$

$$\begin{bmatrix}\tilde{\vec{e}}_1\cdot \\ \tilde{\vec{e}}_2\cdot\end{bmatrix} = \begin{bmatrix}{\tilde{e}^1}_1 & {\tilde{e}^1}_2 \\ {\tilde{e}^2}_1 & {\tilde{e}^2}_2\end{bmatrix}^\intercal \begin{bmatrix}\vec{e}_1\cdot \\ \vec{e}_2\cdot\end{bmatrix} \quad \begin{bmatrix}\tilde{\vec{e}}_1 & \tilde{\vec{e}}_2\end{bmatrix} = \begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix} \begin{bmatrix}{\tilde{e}^1}_1 & {\tilde{e}^1}_2 \\ {\tilde{e}^2}_1 & {\tilde{e}^2}_2\end{bmatrix} \quad \begin{bmatrix}\tilde{v}^1 \\ \tilde{v}^2\end{bmatrix} = \begin{bmatrix}{e^1}_1 & {e^1}_2 \\ {e^2}_1 & {e^2}_2\end{bmatrix} \begin{bmatrix}v^1 \\ v^2\end{bmatrix}$$

$$\begin{bmatrix}\vec{e}_1\cdot \\ \vec{e}_2\cdot\end{bmatrix} = \begin{bmatrix}{e^1}_1 & {e^1}_2 \\ {e^2}_1 & {e^2}_2\end{bmatrix}^\intercal \begin{bmatrix}\tilde{\vec{e}}_1\cdot \\ \tilde{\vec{e}}_2\cdot\end{bmatrix} \quad \begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix} = \begin{bmatrix}\tilde{\vec{e}}_1 & \tilde{\vec{e}}_2\end{bmatrix} \begin{bmatrix}{e^1}_1 & {e^1}_2 \\ {e^2}_1 & {e^2}_2\end{bmatrix} \quad \begin{bmatrix}v^1 \\ v^2\end{bmatrix} = \begin{bmatrix}{\tilde{e}^1}_1 & {\tilde{e}^1}_2 \\ {\tilde{e}^2}_1 & {\tilde{e}^2}_2\end{bmatrix} \begin{bmatrix}\tilde{v}^1 \\ \tilde{v}^2\end{bmatrix}$$

$$\begin{bmatrix}\tilde{\vec{e}}_1\cdot \\ \tilde{\vec{e}}_2\cdot\end{bmatrix} = \begin{bmatrix}\tilde{e}_1^{\intercal 1} & \tilde{e}_1^{\intercal 2} \\ \tilde{e}_2^{\intercal 1} & \tilde{e}_2^{\intercal 2}\end{bmatrix} \begin{bmatrix}\vec{e}_1\cdot \\ \vec{e}_2\cdot\end{bmatrix} \quad \begin{bmatrix}\tilde{\alpha}_1 & \tilde{\alpha}_2\end{bmatrix} = \begin{bmatrix}\alpha_1 & \alpha_2\end{bmatrix} \begin{bmatrix}{\tilde{e}^1}_1 & {\tilde{e}^1}_2 \\ {\tilde{e}^2}_1 & {\tilde{e}^2}_2\end{bmatrix} \quad \begin{bmatrix}\tilde{\overleftarrow{\epsilon}}^1 \\ \tilde{\overleftarrow{\epsilon}}^2\end{bmatrix} = \begin{bmatrix}{e^1}_1 & {e^1}_2 \\ {e^2}_1 & {e^2}_2\end{bmatrix} \begin{bmatrix}\overleftarrow{\epsilon}^1 \\ \overleftarrow{\epsilon}^2\end{bmatrix}$$

$$\begin{bmatrix}\vec{e}_1\cdot \\ \vec{e}_2\cdot\end{bmatrix} = \begin{bmatrix}e_1^{\intercal 1} & e_1^{\intercal 2} \\ e_2^{\intercal 1} & e_2^{\intercal 2}\end{bmatrix} \begin{bmatrix}\tilde{\vec{e}}_1\cdot \\ \tilde{\vec{e}}_2\cdot\end{bmatrix} \quad \begin{bmatrix}\alpha_1 & \alpha_2\end{bmatrix} = \begin{bmatrix}\tilde{\alpha}_1 & \tilde{\alpha}_2\end{bmatrix} \begin{bmatrix}{e^1}_1 & {e^1}_2 \\ {e^2}_1 & {e^2}_2\end{bmatrix} \quad \begin{bmatrix}\overleftarrow{\epsilon}^1 \\ \overleftarrow{\epsilon}^2\end{bmatrix} = \begin{bmatrix}{\tilde{e}^1}_1 & {\tilde{e}^1}_2 \\ {\tilde{e}^2}_1 & {\tilde{e}^2}_2\end{bmatrix} \begin{bmatrix}\tilde{\overleftarrow{\epsilon}}^1 \\ \tilde{\overleftarrow{\epsilon}}^2\end{bmatrix}$$

$$\begin{bmatrix}\tilde{g}_{11} & \tilde{g}_{12} \\ \tilde{g}_{21} & \tilde{g}_{22}\end{bmatrix} = \begin{bmatrix}\tilde{e}_1^{\intercal 1} & \tilde{e}_1^{\intercal 2} \\ \tilde{e}_2^{\intercal 1} & \tilde{e}_2^{\intercal 2}\end{bmatrix} \begin{bmatrix}g_{11} & g_{12} \\ g_{21} & g_{22}\end{bmatrix} \begin{bmatrix}{\tilde{e}^1}_1 & {\tilde{e}^1}_2 \\ {\tilde{e}^2}_1 & {\tilde{e}^2}_2\end{bmatrix} = \begin{bmatrix}{\tilde{e}^1}_1 & {\tilde{e}^1}_2 \\ {\tilde{e}^2}_1 & {\tilde{e}^2}_2\end{bmatrix}^\intercal \begin{bmatrix}\vec{e}_1\cdot \\ \vec{e}_2\cdot\end{bmatrix} \begin{bmatrix}\vec{e}_1 & \vec{e}_2\end{bmatrix} \begin{bmatrix}{\tilde{e}^1}_1 & {\tilde{e}^1}_2 \\ {\tilde{e}^2}_1 & {\tilde{e}^2}_2\end{bmatrix}$$

$$\begin{bmatrix}g_{11} & g_{12} \\ g_{21} & g_{22}\end{bmatrix} = \begin{bmatrix}e_1^{\intercal 1} & e_1^{\intercal 2} \\ e_2^{\intercal 1} & e_2^{\intercal 2}\end{bmatrix} \begin{bmatrix}\tilde{g}_{11} & \tilde{g}_{12} \\ \tilde{g}_{21} & \tilde{g}_{22}\end{bmatrix} \begin{bmatrix}{e^1}_1 & {e^1}_2 \\ {e^2}_1 & {e^2}_2\end{bmatrix} = \begin{bmatrix}{e^1}_1 & {e^1}_2 \\ {e^2}_1 & {e^2}_2\end{bmatrix}^\intercal \begin{bmatrix}\tilde{\vec{e}}_1\cdot \\ \tilde{\vec{e}}_2\cdot\end{bmatrix} \begin{bmatrix}\tilde{\vec{e}}_1 & \tilde{\vec{e}}_2\end{bmatrix} \begin{bmatrix}{e^1}_1 & {e^1}_2 \\ {e^2}_1 & {e^2}_2\end{bmatrix}$$

$$\begin{bmatrix}\tilde{\mathfrak{g}}^{11} & \tilde{\mathfrak{g}}^{12} \\ \tilde{\mathfrak{g}}^{21} & \tilde{\mathfrak{g}}^{22}\end{bmatrix} = \begin{bmatrix}{e^1}_1 & {e^1}_2 \\ {e^2}_1 & {e^2}_2\end{bmatrix} \begin{bmatrix}\mathfrak{g}^{11} & \mathfrak{g}^{12} \\ \mathfrak{g}^{21} & \mathfrak{g}^{22}\end{bmatrix} \begin{bmatrix}e_1^{\intercal 1} & e_1^{\intercal 2} \\ e_2^{\intercal 1} & e_2^{\intercal 2}\end{bmatrix} = \begin{bmatrix}{e^1}_1 & {e^1}_2 \\ {e^2}_1 & {e^2}_2\end{bmatrix} \begin{bmatrix}\mathfrak{g}^{11} & \mathfrak{g}^{12} \\ \mathfrak{g}^{21} & \mathfrak{g}^{22}\end{bmatrix} \begin{bmatrix}{e^1}_1 & {e^1}_2 \\ {e^2}_1 & {e^2}_2\end{bmatrix}^\intercal$$

$$\begin{bmatrix}\mathfrak{g}^{11} & \mathfrak{g}^{12} \\ \mathfrak{g}^{21} & \mathfrak{g}^{22}\end{bmatrix} = \begin{bmatrix}{\tilde{e}^1}_1 & {\tilde{e}^1}_2 \\ {\tilde{e}^2}_1 & {\tilde{e}^2}_2\end{bmatrix} \begin{bmatrix}\tilde{\mathfrak{g}}^{11} & \tilde{\mathfrak{g}}^{12} \\ \tilde{\mathfrak{g}}^{21} & \tilde{\mathfrak{g}}^{22}\end{bmatrix} \begin{bmatrix}\tilde{e}_1^{\intercal 1} & \tilde{e}_1^{\intercal 2} \\ \tilde{e}_2^{\intercal 1} & \tilde{e}_2^{\intercal 2}\end{bmatrix} = \begin{bmatrix}{\tilde{e}^1}_1 & {\tilde{e}^1}_2 \\ {\tilde{e}^2}_1 & {\tilde{e}^2}_2\end{bmatrix} \begin{bmatrix}\tilde{\mathfrak{g}}^{11} & \tilde{\mathfrak{g}}^{12} \\ \tilde{\mathfrak{g}}^{21} & \tilde{\mathfrak{g}}^{22}\end{bmatrix} \begin{bmatrix}{\tilde{e}^1}_1 & {\tilde{e}^1}_2 \\ {\tilde{e}^2}_1 & {\tilde{e}^2}_2\end{bmatrix}^\intercal$$

$$\begin{bmatrix}{\tilde{L}^1}_1 & {\tilde{L}^1}_2 \\ {\tilde{L}^2}_1 & {\tilde{L}^2}_2\end{bmatrix} = \begin{bmatrix}{e^1}_1 & {e^1}_2 \\ {e^2}_1 & {e^2}_2\end{bmatrix} \begin{bmatrix}{L^1}_1 & {L^1}_2 \\ {L^2}_1 & {L^2}_2\end{bmatrix} \begin{bmatrix}{\tilde{e}^1}_1 & {\tilde{e}^1}_2 \\ {\tilde{e}^2}_1 & {\tilde{e}^2}_2\end{bmatrix} = \left(\begin{bmatrix}\tilde{e}_1^{\intercal 1} & \tilde{e}_1^{\intercal 2} \\ \tilde{e}_2^{\intercal 1} & \tilde{e}_2^{\intercal 2}\end{bmatrix} \begin{bmatrix}L_1^{\intercal 1} & L_1^{\intercal 2} \\ L_2^{\intercal 1} & L_2^{\intercal 2}\end{bmatrix} \begin{bmatrix}e_1^{\intercal 1} & e_1^{\intercal 2} \\ e_2^{\intercal 1} & e_2^{\intercal 2}\end{bmatrix}\right)^\intercal$$

$$\begin{bmatrix}{L^1}_1 & {L^1}_2 \\ {L^2}_1 & {L^2}_2\end{bmatrix} = \begin{bmatrix}{\tilde{e}^1}_1 & {\tilde{e}^1}_2 \\ {\tilde{e}^2}_1 & {\tilde{e}^2}_2\end{bmatrix} \begin{bmatrix}{\tilde{L}^1}_1 & {\tilde{L}^1}_2 \\ {\tilde{L}^2}_1 & {\tilde{L}^2}_2\end{bmatrix} \begin{bmatrix}{e^1}_1 & {e^1}_2 \\ {e^2}_1 & {e^2}_2\end{bmatrix} = \left(\begin{bmatrix}e_1^{\intercal 1} & e_1^{\intercal 2} \\ e_2^{\intercal 1} & e_2^{\intercal 2}\end{bmatrix} \begin{bmatrix}\tilde{L}_1^{\intercal 1} & \tilde{L}_1^{\intercal 2} \\ \tilde{L}_2^{\intercal 1} & \tilde{L}_2^{\intercal 2}\end{bmatrix} \begin{bmatrix}\tilde{e}_1^{\intercal 1} & \tilde{e}_1^{\intercal 2} \\ \tilde{e}_2^{\intercal 1} & \tilde{e}_2^{\intercal 2}\end{bmatrix}\right)^\intercal$$

$$\begin{bmatrix}\tilde{L}_1^{\intercal 1} & \tilde{L}_1^{\intercal 2} \\ \tilde{L}_2^{\intercal 1} & \tilde{L}_2^{\intercal 2}\end{bmatrix} = \begin{bmatrix}{\tilde{L}^1}_1 & {\tilde{L}^1}_2 \\ {\tilde{L}^2}_1 & {\tilde{L}^2}_2\end{bmatrix}^\intercal = \begin{bmatrix}{\tilde{e}^1}_1 & {\tilde{e}^1}_2 \\ {\tilde{e}^2}_1 & {\tilde{e}^2}_2\end{bmatrix}^\intercal \begin{bmatrix}{L^1}_1 & {L^1}_2 \\ {L^2}_1 & {L^2}_2\end{bmatrix}^\intercal \begin{bmatrix}{e^1}_1 & {e^1}_2 \\ {e^2}_1 & {e^2}_2\end{bmatrix}^\intercal = \begin{bmatrix}\tilde{e}_1^{\intercal 1} & \tilde{e}_1^{\intercal 2} \\ \tilde{e}_2^{\intercal 1} & \tilde{e}_2^{\intercal 2}\end{bmatrix} \begin{bmatrix}L_1^{\intercal 1} & L_1^{\intercal 2} \\ L_2^{\intercal 1} & L_2^{\intercal 2}\end{bmatrix} \begin{bmatrix}e_1^{\intercal 1} & e_1^{\intercal 2} \\ e_2^{\intercal 1} & e_2^{\intercal 2}\end{bmatrix}$$

$$\begin{bmatrix}L_1^{\intercal 1} & L_1^{\intercal 2} \\ L_2^{\intercal 1} & L_2^{\intercal 2}\end{bmatrix} = \begin{bmatrix}{L^1}_1 & {L^1}_2 \\ {L^2}_1 & {L^2}_2\end{bmatrix}^\intercal = \begin{bmatrix}{e^1}_1 & {e^1}_2 \\ {e^2}_1 & {e^2}_2\end{bmatrix}^\intercal \begin{bmatrix}{\tilde{L}^1}_1 & {\tilde{L}^1}_2 \\ {\tilde{L}^2}_1 & {\tilde{L}^2}_2\end{bmatrix}^\intercal \begin{bmatrix}{\tilde{e}^1}_1 & {\tilde{e}^1}_2 \\ {\tilde{e}^2}_1 & {\tilde{e}^2}_2\end{bmatrix}^\intercal = \begin{bmatrix}e_1^{\intercal 1} & e_1^{\intercal 2} \\ e_2^{\intercal 1} & e_2^{\intercal 2}\end{bmatrix} \begin{bmatrix}\tilde{L}_1^{\intercal 1} & \tilde{L}_1^{\intercal 2} \\ \tilde{L}_2^{\intercal 1} & \tilde{L}_2^{\intercal 2}\end{bmatrix} \begin{bmatrix}\tilde{e}_1^{\intercal 1} & \tilde{e}_1^{\intercal 2} \\ \tilde{e}_2^{\intercal 1} & \tilde{e}_2^{\intercal 2}\end{bmatrix}$$

Table 1.4: transformation as ordered tensor

review of dual space, in physics,

inner product	metric / norm	
$\langle\cdot\vert\cdot\rangle = \langle\square\vert\square\rangle : \mathcal{V}^2 \to \mathbb{R}$	$\eta = \eta(\square,\square) : \mathcal{V}^2 \to \mathbb{F} \in \{\mathbb{C},\mathbb{R},\cdots\}$	function (f)
$\left\langle\vec{v}\vert\vec{v}\right\rangle \geq 0 \quad , \left\langle\vec{v}\vert\vec{v}\right\rangle = 0 \Leftrightarrow \vec{v} = \vec{0}$	$\left\Vert\vec{v}\right\Vert^2 = \eta\left(\vec{v},\vec{v}\right) \gtreqless 0$	positivity (p)
$\left\langle\vec{u}\vert\vec{v}\right\rangle = \left\langle\vec{v}\vert\vec{u}\right\rangle^*$	$\eta\left(\vec{u},\vec{v}\right) = \eta\left(\vec{v},\vec{u}\right)$	symmetry(s)
$\left\langle\vec{u}\vert s\vec{v}\right\rangle = s\left\langle\vec{u}\vert\vec{v}\right\rangle$	$s\eta\left(\vec{u},\vec{v}\right) = \eta\left(s\vec{u},\vec{v}\right) = \eta\left(\vec{u},s\vec{v}\right)$	homogeneity (h)
$\left\langle\vec{u}\vert\vec{v}+\vec{w}\right\rangle = \left\langle\vec{u}\vert\vec{v}\right\rangle + \left\langle\vec{u}\vert\vec{w}\right\rangle$	$\left\Vert\vec{u}+\vec{v}\right\Vert \gtreqless \left\Vert\vec{u}\right\Vert + \left\Vert\vec{v}\right\Vert$	additivity (a)

dimension

$$\dim \mathcal{V} = \arg\max_n \left| \left\{ \vec{e}_i \middle| \vec{e}_i v^i = \vec{0} \Rightarrow (\cdots, v^i, \cdots) = (\cdots, 0, \cdots) \right\}_{i=1}^{n} \right|$$

linearity

$$\begin{aligned} &\phi \text{ is linear} \\ \Leftrightarrow &\begin{cases} \phi : D \to C & \Leftrightarrow \phi \in C^D \\ \phi(s_x x + s_y y) = s_x \phi(x) + s_y \phi(y) & \forall s_x x, s_y y \in D, \exists s_x \phi(x), s_y \phi(y) \in C \end{cases} \end{aligned}$$

antilinearity

$$\begin{aligned} &\psi \text{ is antilinear} \\ \Leftrightarrow &\begin{cases} \psi : D \to C & \Leftrightarrow \phi \in C^D \\ s_x^* \text{ is a conjugate of } s_x & \forall s_x \\ \psi(s_x x + s_y y) = s_x^* \psi(x) + s_y^* \psi(y) & \forall s_x x, s_y y \in D, \exists s_x^* \psi(x), s_y^* \psi(y) \in C \end{cases} \end{aligned}$$

vector space

$$\mathcal{V}$$

vector basis

$$\left\{ \vec{e}_i \middle| v^i v^i = 0 \left[\vec{e}_i v^i \neq \vec{0} \right] \right\}_{i=1}^{\dim \mathcal{V}} \subseteq \mathcal{V}$$

dual space

$$\forall \overleftarrow{\alpha} \in \mathcal{V}^*, \forall \vec{v} \in \mathcal{V} \left[\overleftarrow{\alpha}\left(\vec{v}\right) \in \mathbb{F} \Leftrightarrow \mathcal{V}^* = \mathbb{F}^{\mathcal{V}} \Leftrightarrow \mathcal{V}^* : \mathcal{V} \to \mathbb{F} \right]$$

dual basis

$$\overleftarrow{\epsilon}^i \left(\vec{e}_j\right) = \overleftarrow{\epsilon}^i \vec{e}_j = \delta^i{}_j \in \mathbb{F}$$

$$\left\{ \overleftarrow{\epsilon}^i \middle| \overleftarrow{\epsilon}^i \left(\vec{e}_j\right) = \overleftarrow{\epsilon}^i \vec{e}_j = \delta^i{}_j \in \mathbb{F} \right\}_{i=1}^{\dim \mathcal{V}} \subseteq \mathcal{V}^*$$

$$\begin{aligned} \overleftarrow{\alpha}\left(\vec{v}\right) = \overleftarrow{\alpha}\vec{v} &= \overleftarrow{\alpha}\left(\vec{e}_i v^i\right) && \text{vector basis decomposition} \\ &= \overleftarrow{\alpha}\left(\vec{e}_i\right) v^i && \text{linearity} \\ &= \overleftarrow{\alpha}\left(\vec{e}_i\right)\left(\vec{e}_i \cdot \vec{v}\right) && v^i = \vec{e}_i \cdot \vec{v} \\ &= \alpha_i \left(\vec{e}_i \cdot \vec{v}\right) && \alpha_i = \overleftarrow{\alpha}\left(\vec{e}_i\right) \\ &= \left(\alpha_i \vec{e}_i \cdot\right) \vec{v} && \overleftarrow{\epsilon}^i = \vec{e}_i \cdot \\ &= \left(\alpha_i \overleftarrow{\epsilon}^i\right) \vec{v} = \alpha_i \overleftarrow{\epsilon}^i \left(\vec{v}\right) \\ \overleftarrow{\alpha} &= \alpha_i \overleftarrow{\epsilon}^i && \text{where } \overleftarrow{\epsilon}^i = \vec{e}_i \cdot \end{aligned}$$

$$\begin{aligned} \overleftarrow{\alpha} &= \alpha_i \overleftarrow{\epsilon}^i \\ \overleftarrow{\alpha}\left(\vec{e}_j\right) = \overleftarrow{\alpha}\vec{e}_j &= \alpha_i \overleftarrow{\epsilon}^i \vec{e}_j && \overleftarrow{\alpha}\left(\vec{e}_j\right) = \overleftarrow{\alpha}\vec{e}_j \\ \overleftarrow{\alpha}\left(\vec{e}_j\right) = \overleftarrow{\alpha}\vec{e}_j &= \alpha_i \delta^i{}_j && \overleftarrow{\epsilon}^i \vec{e}_j = \delta^i{}_j \\ \alpha_j = \overleftarrow{\alpha}\left(\vec{e}_j\right) = \overleftarrow{\alpha}\vec{e}_j &= \alpha_j && \alpha_j = \overleftarrow{\alpha}\left(\vec{e}_j\right) \Leftarrow \alpha_i = \overleftarrow{\alpha}\left(\vec{e}_i\right) \\ & && \alpha_j = \alpha_i \delta^i{}_j \end{aligned}$$

$$\dim \mathcal{V}^* = \dim \mathcal{V} \Rightarrow \mathcal{V}^* \cong \mathcal{V}$$

but also able to choose other dual basis

$$\left\{ \overleftarrow{\tilde{\epsilon}}^i \middle| \overleftarrow{\tilde{\epsilon}}^i \left(\vec{e}_j\right) = \overleftarrow{\tilde{\epsilon}}^i \vec{e}_j = \tilde{\delta}^i{}_j \in \mathbb{F} \right\}_{i=1}^{\dim \mathcal{V}} \subseteq \mathcal{V}^*$$

double dual space

$$\forall v^{**} \in \mathcal{V}^{**}, \forall \overleftarrow{\alpha} \in \mathcal{V}^* \left[v^{**}\left(\overleftarrow{\alpha}\right) \in \mathbb{F} \Leftrightarrow \mathcal{V}^{**} = \mathbb{F}^{\mathcal{V}^*} \Leftrightarrow \mathcal{V}^{**} : \mathcal{V}^* \to \mathbb{F} \right]$$

natural correspondence

$$v^{**}\left(\overleftarrow{\alpha}\right) \in \mathbb{F} \overset{\text{natural choice}}{\longrightarrow} v^{**}\left(\overleftarrow{\alpha}\right) = \overleftarrow{\alpha}\left(\vec{v}\right) \in \mathbb{F}$$

$$v^{**}\left(\overleftarrow{\alpha}\right) = \Box\vec{v}_{\Box=\overleftarrow{\alpha}} = \overleftarrow{\alpha}\vec{v} = \overleftarrow{\alpha}\Box_{\Box=\vec{v}} = \overleftarrow{\alpha}\left(\vec{v}\right)$$

$$v^{**}\left(\overleftarrow{\alpha}\right) = \overleftarrow{\alpha}\vec{v} = \overleftarrow{\alpha}\left(\vec{v}\right)$$

$$\left(\overleftarrow{\alpha}\right)\vec{v} = \overleftarrow{\alpha}v^{**} = \overleftarrow{\alpha}\left(v^{**}\right)$$

$$\begin{array}{ccc} \mathbb{F} \ni v^{**}\left(\overleftarrow{\alpha}\right) & = \overleftarrow{\alpha}\vec{v} = & \overleftarrow{\alpha}\left(\vec{v}\right) \in \mathbb{F} \\ \updownarrow & \updownarrow & \updownarrow \\ \left(\overleftarrow{\alpha}\right) v^{**} & = \overleftarrow{\alpha}v^{**} = & \left(\overleftarrow{\alpha}\right)\vec{v} \end{array}$$

$$\dim \mathcal{V}^{**} = \dim \mathcal{V}^{*} = \dim \mathcal{V} \Rightarrow \mathcal{V}^{**} \cong \mathcal{V}^{*} \cong \mathcal{V}$$

natural isomorphism

$$\begin{cases} \mathcal{V}^{**} \cong \mathcal{V} & \text{isomorphism} \\ v^{**} \in \mathcal{V}^{**} & \\ \overleftarrow{\alpha} \in \mathcal{V}^{*} & \\ \vec{v} \in \mathcal{V} & \\ \begin{array}{ccc} \mathbb{F} \ni v^{**}\left(\overleftarrow{\alpha}\right) & = \overleftarrow{\alpha}\vec{v} = & \overleftarrow{\alpha}\left(\vec{v}\right) \in \mathbb{F} \\ \updownarrow & \updownarrow & \updownarrow \\ \left(\overleftarrow{\alpha}\right) v^{**} & = \overleftarrow{\alpha}v^{**} = & \left(\overleftarrow{\alpha}\right)\vec{v} \end{array} & \text{natural correspondence} \end{cases} \Rightarrow \mathcal{V}^{**} = \mathcal{V}$$

complex conjugate

$$\begin{aligned} zw &= \left(z^1 + \mathrm{i}z^2\right)\left(w^1 + \mathrm{i}w^2\right) \\ &= \left(z^1w^1 - z^2w^2\right) + \mathrm{i}\left(z^1w^2 + z^2w^1\right) \end{aligned}$$

$$\begin{aligned} zz &= \left(z^1 + \mathrm{i}z^2\right)\left(z^1 + \mathrm{i}z^2\right) \\ &= \left(z^1z^1 - z^2z^2\right) + \mathrm{i}\left(z^1z^2 + z^2z^1\right) \\ &= z^{12} - z^{22} + \mathrm{i}2z^1z^2 && z^1z^2 = z^2z^1 \end{aligned}$$

$$\begin{aligned} z^*w &= \left(z^1 - \mathrm{i}z^2\right)\left(w^1 + \mathrm{i}w^2\right) \\ &= \left(z^1w^1 + z^2w^2\right) + \mathrm{i}\left(z^1w^2 - z^2w^1\right) \end{aligned}$$

$$\begin{aligned} z^*z &= \left(z^1 - \mathrm{i}z^2\right)\left(z^1 + \mathrm{i}z^2\right) \\ &= \left(z^1z^1 + z^2z^2\right) + \mathrm{i}\left(z^1z^2 - z^2z^1\right) \\ &= z^{12} + z^{22} + \mathrm{i}0 && z^1z^2 = z^2z^1 \\ &= z^{12} + z^{22} = |z|^2 && |z|^2 = \left|z^1 + \mathrm{i}z^2\right|^2 = z^{12} + z^{22} \end{aligned}$$

$$\begin{aligned} \begin{bmatrix} z_1 & z_2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} &= z_1w_1 + z_2w_2 \\ &= \left(z_1^1 + \mathrm{i}z_1^2\right)\left(w_1^1 + \mathrm{i}w_1^2\right) + \left(z_2^1 + \mathrm{i}z_2^2\right)\left(w_2^1 + \mathrm{i}w_2^2\right) \\ &= \left(z_1^1w_1^1 - z_1^2w_1^2\right) + \mathrm{i}\left(z_1^1w_1^2 + z_1^2w_1^1\right) + \left(z_2^1w_2^1 - z_2^2w_2^2\right) + \mathrm{i}\left(z_2^1w_2^2 + z_2^2w_2^1\right) \\ &= \left(z_1^1w_1^1 - z_1^2w_1^2 + z_2^1w_2^1 - z_2^2w_2^2\right) + \mathrm{i}\left(z_1^1w_1^2 + z_1^2w_1^1 + z_2^1w_2^2 + z_2^2w_2^1\right) \end{aligned}$$

$$\begin{aligned} \begin{bmatrix} z_1^* & z_2^* \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} &= z_1^*w_1 + z_2^*w_2 \\ &= \left(z_1^1 - \mathrm{i}z_1^2\right)\left(w_1^1 + \mathrm{i}w_1^2\right) + \left(z_2^1 - \mathrm{i}z_2^2\right)\left(w_2^1 + \mathrm{i}w_2^2\right) \\ &= \left(z_1^1w_1^1 + z_1^2w_1^2\right) + \mathrm{i}\left(z_1^1w_1^2 - z_1^2w_1^1\right) + \left(z_2^1w_2^1 + z_2^2w_2^2\right) + \mathrm{i}\left(z_2^1w_2^2 - z_2^2w_2^1\right) \\ &= \left(z_1^1w_1^1 + z_1^2w_1^2 + z_2^1w_2^1 + z_2^2w_2^2\right) + \mathrm{i}\left(z_1^1w_1^2 - z_1^2w_1^1 + z_2^1w_2^2 - z_2^2w_2^1\right) \end{aligned}$$

$$\begin{bmatrix} z_1^* & z_2^* \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = z_1^* z_1 + z_2^* z_2$$
$$= \left(z_1^1 - \mathrm{i} z_1^2\right)\left(z_1^1 + \mathrm{i} z_1^2\right) + \left(z_2^1 - \mathrm{i} z_2^2\right)\left(z_2^1 + \mathrm{i} z_2^2\right)$$
$$= \left(z_1^1 z_1^1 + z_1^2 z_1^2\right) + \mathrm{i}\left(z_1^1 z_1^2 - z_1^2 z_1^1\right) + \left(z_2^1 z_2^1 + z_2^2 z_2^2\right) + \mathrm{i}\left(z_2^1 z_2^2 - z_2^2 z_2^1\right)$$
$$= \left(z_1^{12} + z_1^{22} + z_2^{12} + z_2^{22}\right) + \mathrm{i}\left(0 + 0\right)$$
$$= |z_1|^2 + |z_2|^2$$

$$\begin{bmatrix} z_1^* & z_2^* \end{bmatrix} = \left(\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}^*\right)^\intercal = \left(\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}^\intercal\right)^* = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}^\dagger$$

$$\left|\vec{v}\right\rangle \equiv v \in \mathcal{V} \leftrightarrow v^j = \begin{bmatrix} v^1 \\ v^2 \end{bmatrix}$$

in inner product space,

$$\left\langle\vec{v}\right| \equiv v^\dagger \equiv v^* \in \mathcal{V}^* \leftrightarrow \left(v^\dagger\right)_i \equiv \left(v^*\right)_i \overset{\text{e.g.}}{=} \begin{bmatrix} (v^*)_1 & (v^*)_2 \end{bmatrix} = \begin{bmatrix} (v^1)^* & (v^2)^* \end{bmatrix} = \begin{bmatrix} v^{1*} & v^{2*} \end{bmatrix} = \begin{bmatrix} v^1 & v^2 \end{bmatrix}^* = \left(v^i\right)^*$$

$$\left\langle\vec{v}\right| \equiv v^\dagger \equiv v^* \leftrightarrow \left(v^\dagger\right)_i \equiv \left(v^*\right)_i = \left(v^i\right)^*$$
$$\leftrightarrow v_i \equiv v_i = \left(v^i\right)^* \qquad \text{omit dagger } \dagger \because v_i \leftrightarrow v^\dagger \equiv v^* \in \mathcal{V}^*$$

in metric space,

$$\left\langle\vec{v}\right| \equiv v^\dagger \equiv v^* \in \mathcal{V}^* \leftrightarrow \left(v^\dagger\right)_i \equiv \left(v^*\right)_i \overset{\text{e.g.}}{=} \begin{bmatrix} (v^*)_1 & (v^*)_2 \end{bmatrix} = \begin{bmatrix} v^1 & v^2 \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = v^j g_{ji}$$

$$\left\langle\vec{v}\right| \equiv v^\dagger \equiv v^* \leftrightarrow \left(v^\dagger\right)_i \equiv \left(v^*\right)_i = v^j g_{ji}$$
$$\leftrightarrow v_i \equiv v_i = v^j g_{ji} \qquad \text{omit dagger } \dagger \because v_i \leftrightarrow v^\dagger \equiv v^* \in \mathcal{V}^*$$

$$A \overset{\text{e.g.}}{=} \begin{bmatrix} A^1{}_1 & A^1{}_2 \\ A^2{}_1 & A^2{}_2 \end{bmatrix} \leftrightarrow A^i{}_j$$

transpose

$$L^i{}_j^\intercal = \left(L^i{}_j\right)^\intercal = L^j{}_i = L^{\intercal j}_i = \left(L^\intercal\right)_i{}^j$$
$$\left(A^i{}_j\right)^\intercal = A^i{}_j^\intercal \leftrightarrow A^\intercal \overset{\text{e.g.}}{=} \begin{bmatrix} A^1{}_1 & A^1{}_2 \\ A^2{}_1 & A^2{}_2 \end{bmatrix}^\intercal = \begin{bmatrix} A^1{}_1 & A^2{}_1 \\ A^1{}_2 & A^2{}_2 \end{bmatrix} \leftrightarrow A^j{}_i = A^{\intercal j}_i = \left(A^\intercal\right)_i{}^j \leftrightarrow \begin{bmatrix} A^{\intercal 1}_1 & A^{\intercal 2}_1 \\ A^{\intercal 1}_2 & A^{\intercal 2}_2 \end{bmatrix}$$
$$A^j{}_i = A^{\intercal j}_i$$
$$A^j{}_i \underset{A^\intercal_{i\rightarrow}{}^{\leftarrow j}}{\overset{A^{j\rightarrow}{}_{\leftarrow i}}{\rightleftharpoons}} A^{\intercal j}_i$$

in inner product space, conjugate transpose

$$\left(A^\dagger\right)^i{}_j = A^{\dagger i}{}_j \leftrightarrow A^\dagger \equiv \left(A^\intercal\right)^* = \left(A^*\right)^\intercal$$

$$\left(A^\dagger\right)^i{}_j = A^{\dagger i}{}_j \leftrightarrow A^\dagger \equiv \begin{cases} \left(A^\intercal\right)^* & \leftrightarrow \left(A^{\intercal j}_i\right)^* = \left(A^j{}_i\right)^* = A^{j*}_i \\ \left(A^*\right)^\intercal & \leftrightarrow \left(\left(A^i{}_j\right)^*\right)^\intercal = \left(A^{i*}_j\right)^\intercal = A^{j*}_i \end{cases}$$

$$\begin{bmatrix} (A^\dagger)^1{}_1 & (A^\dagger)^1{}_2 \\ (A^\dagger)^2{}_1 & (A^\dagger)^2{}_2 \end{bmatrix} = \begin{bmatrix} A^{\dagger 1}{}_1 & A^{\dagger 1}{}_2 \\ A^{\dagger 2}{}_1 & A^{\dagger 2}{}_2 \end{bmatrix} = \begin{cases} \begin{bmatrix} A^1{}_1 & A^2{}_1 \\ A^1{}_2 & A^2{}_2 \end{bmatrix}^* & = \begin{bmatrix} A^{1*}_1 & A^{2*}_1 \\ A^{1*}_2 & A^{2*}_2 \end{bmatrix} \\ \begin{bmatrix} A^{1*}_1 & A^{1*}_2 \\ A^{2*}_1 & A^{2*}_2 \end{bmatrix}^\intercal & = \begin{bmatrix} A^{1*}_1 & A^{2*}_1 \\ A^{1*}_2 & A^{2*}_2 \end{bmatrix} \end{cases}$$

$$A_j{}^i \overset{\text{def.}}{=} A^{j*}_i$$
$$A_j{}^i \equiv A^{j*}_i$$
$$A_j{}^i \underset{A^{\,j}_{\downarrow}{}^*_{\uparrow i}}{\overset{A_{\uparrow j}{}^{i}_{\downarrow}}{\rightleftharpoons}} A^{j*}_i$$
$$A^{\dagger i}{}_j = A^{j*}_i$$

$$A_j{}^i \equiv A^{j*}_{\ i} = A^{\dagger i}{}_j$$
$$A_j{}^i = A^{\dagger i}{}_j$$

in metric space,

$$\begin{aligned}
(A^\dagger)^i{}_j = A^{\dagger i}{}_j \leftrightarrow A^\dagger &\equiv \mathfrak{g} A^\intercal g \\
&\leftrightarrow \mathfrak{g}^{ih} A^{\intercal k}_h g_{kj} \\
&= \mathfrak{g}^{ih} A^k{}_h g_{kj} && A^j{}_i = A^{\intercal j}_i \\
&= g_{kj} A^k{}_h \mathfrak{g}^{ih} \\
&= g_{jk} A^k{}_h \mathfrak{g}^{hi} && g_{jk} = g_{kj} \\
& && \mathfrak{g}^{hi} = \mathfrak{g}^{ih} \\
&= A_j{}^i && g_{jk} \text{ lowering index(ies)} \\
& && \mathfrak{g}^{hi} \text{ raising index(ies)} \\
A^{\dagger i}{}_j &= A_j{}^i \\
A_j{}^i &= A^{\dagger i}{}_j
\end{aligned}$$

$$\begin{cases} A_j{}^i = A^{\dagger i}{}_j \equiv A^{j*}_{\ i} & \text{inner product space} \\ A_j{}^i = A^{\dagger i}{}_j \equiv \mathfrak{g}^{ih} A^{\intercal k}_h g_{kj} & \text{metric space} \end{cases} \Rightarrow A_j{}^i \equiv A^{\dagger i}{}_j$$

$$\begin{cases}
\left|\vec{v}\right\rangle = v & \leftrightarrow v^j \\
\left\langle\vec{v}\right| = v^\dagger & \leftrightarrow v_i = \begin{cases} (v^i)^* & \text{inner product space} \\ v^j g_{ji} & \text{metric space} \end{cases} \\
A & \leftrightarrow A^i{}_j \\
A^\intercal & \leftrightarrow A^{\intercal j}_i \\
A^\dagger & \leftrightarrow A^{\dagger i}{}_j \\
A^j{}_i & = A^{\intercal j}_i \Leftrightarrow A^j{}_i \underset{A^\intercal_{i\rightarrow}\leftarrow j}{\overset{A^{j\rightarrow}\leftarrow i}{\rightleftharpoons}} A^{\intercal j}_i \\
A_j{}^i & = A^{\dagger i}{}_j = \begin{cases} A^{j*}_{\ i} = (A^{\intercal j}_i)^* \Leftrightarrow A_j{}^i \underset{A^{j}_{\downarrow}{}^{*}_{\uparrow i}}{\overset{A_{\uparrow j}{}^{i}_{\downarrow}}{\rightleftharpoons}} A^{j*}_{\ i} & \text{inner product space} \\ g_{jk} A^k{}_h \mathfrak{g}^{hi} = \mathfrak{g}^{ih} A^{\intercal k}_h g_{kj} & \text{metric space} \end{cases} \\
A^i{}_j & = \begin{cases} (A_i{}^j)^* & \text{conjuage transpose in inner product space} \\ \mathfrak{g}^{ih} A_h{}^k g_{kj} & \left.\begin{matrix} g_{jk} \text{ lowering index(ies)} \\ \mathfrak{g}^{hi} \text{ raising index(ies)} \end{matrix}\right\} \text{in metric space} \end{cases}
\end{cases}$$

$$v_i A^{\dagger i}{}_j = (v^i)^* A^{j*}_{\ i} = A^{j*}_{\ i} (v^i)^* = (A^j{}_i v^i)^* = (w^j)^* = w_j$$
$$i \leftrightarrow j$$
$$A^i{}_j v^j = w^i \overset{j \leftrightarrow i}{\Leftrightarrow} A^j{}_i v^i = w^j \Leftrightarrow v_i A^{\dagger i}{}_j = w_j \leftrightarrow A\left|\vec{v}\right\rangle = \left|\vec{w}\right\rangle \Leftrightarrow \left\langle\vec{v}\right| A^\dagger = \left\langle\vec{w}\right|$$

$$v_i A^{\dagger i}{}_j = v^\ell g_{\ell i} g_{jk} A^k{}_h \mathfrak{g}^{hi} = v^\ell g_{\ell i} \mathfrak{g}^{hi} g_{jk} A^k{}_h = v^\ell g_{\ell i} \mathfrak{g}^{ih} g_{jk} A^k{}_h = v^\ell \delta_\ell{}^h g_{jk} A^k{}_h = v^h g_{jk} A^k{}_h = g_{jk} A^k{}_h v^h = g_{jk} w^k = w_j$$
$$i \leftrightarrow k$$
$$A^i{}_j v^j = w^i \overset{j \leftrightarrow h}{\Leftrightarrow} A^k{}_h v^h = w^k \Leftrightarrow v_i A^{\dagger i}{}_j = w_j \leftrightarrow A\left|\vec{v}\right\rangle = \left|\vec{w}\right\rangle \Leftrightarrow \left\langle\vec{v}\right| A^\dagger = \left\langle\vec{w}\right|$$
$$\left\langle\vec{v}\right| A^\intercal = \left\langle\vec{w}\right| \not\Leftrightarrow A\left|\vec{v}\right\rangle = \left|\vec{w}\right\rangle \Leftrightarrow \left\langle\vec{v}\right| A^\dagger = \left\langle\vec{w}\right|$$

multilinear map

$$T$$

rank-0 tensor

$$s \in \mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \cdots\} \qquad \text{0-tensor}$$
$$\binom{0}{0} = 1$$

rank-1 tensor

$$\begin{aligned}
T^i &\leftrightarrow T^i \vec{e}_i && \in \mathcal{V} && (1,0)\text{-tensor} \\
T_i &\leftrightarrow T_i \overleftarrow{\epsilon}^i && \in \mathcal{V}^* && (0,1)\text{-tensor}
\end{aligned}$$

$$\binom{1}{0} + \binom{1}{1} = 1 + 1 = 2$$

rank-2 tensor

$T^{ij} \leftrightarrow T^{ij}\vec{e}_i \otimes \vec{e}_j$	$\in \mathcal{V} \otimes \mathcal{V}$		$(2,0)$-tensor
$T^{i}{}_{j} \leftrightarrow T^{i}{}_{j}\vec{e}_i \otimes \overleftarrow{\epsilon}^j$	$\in \mathcal{V} \otimes \mathcal{V}^*$		$(1,1)$-tensor
$T_{j}{}^{i} \leftrightarrow T_{j}{}^{i}\overleftarrow{\epsilon}^j \otimes \vec{e}_i$	$\in \mathcal{V}^* \otimes \mathcal{V}$	transpose	$(1,1)$-tensor
$T_{ij} \leftrightarrow T_{ij}\overleftarrow{\epsilon}^i \otimes \overleftarrow{\epsilon}^j$	$\in \mathcal{V}^* \otimes \mathcal{V}^*$		$(0,2)$-tensor

$$\binom{2}{0} + \binom{2}{1} + \binom{2}{2} = 1 + 2 + 1 = 4$$

rank-3 tensor

$T^{ijk} \leftrightarrow T^{ijk}\vec{e}_i \otimes \vec{e}_j \otimes \vec{e}_k$	$\in \mathcal{V} \otimes \mathcal{V} \otimes \mathcal{V}$	$(3,0)$-tensor
$T^{ij}{}_{k} \leftrightarrow T^{ij}{}_{k}\vec{e}_i \otimes \vec{e}_j \otimes \overleftarrow{\epsilon}^k$	$\in \mathcal{V} \otimes \mathcal{V} \otimes \mathcal{V}^*$	$(2,1)$-tensor
$T^{i}{}_{j}{}^{k} \leftrightarrow T^{i}{}_{j}{}^{k}\vec{e}_i \otimes \overleftarrow{\epsilon}^j \otimes \vec{e}_k$	$\in \mathcal{V} \otimes \mathcal{V}^* \otimes \mathcal{V}$	$(2,1)$-tensor
$T_{i}{}^{jk} \leftrightarrow T_{i}{}^{jk}\overleftarrow{\epsilon}^i \otimes \vec{e}_j \otimes \vec{e}_k$	$\in \mathcal{V}^* \otimes \mathcal{V} \otimes \mathcal{V}$	$(2,1)$-tensor
$T^{i}{}_{jk} \leftrightarrow T^{i}{}_{jk}\vec{e}_i \otimes \overleftarrow{\epsilon}^j \otimes \overleftarrow{\epsilon}^k$	$\in \mathcal{V} \otimes \mathcal{V}^* \otimes \mathcal{V}^*$	$(1,2)$-tensor
$T_{i}{}^{j}{}_{k} \leftrightarrow T_{i}{}^{j}{}_{k}\overleftarrow{\epsilon}^i \otimes \vec{e}_j \otimes \overleftarrow{\epsilon}^k$	$\in \mathcal{V}^* \otimes \mathcal{V} \otimes \mathcal{V}^*$	$(1,2)$-tensor
$T_{ij}{}^{k} \leftrightarrow T_{ij}{}^{k}\overleftarrow{\epsilon}^i \otimes \overleftarrow{\epsilon}^j \otimes \vec{e}_k$	$\in \mathcal{V}^* \otimes \mathcal{V}^* \otimes \mathcal{V}$	$(1,2)$-tensor
$T_{ijk} \leftrightarrow T_{ijk}\overleftarrow{\epsilon}^i \otimes \overleftarrow{\epsilon}^j \otimes \overleftarrow{\epsilon}^k$	$\in \mathcal{V}^* \otimes \mathcal{V}^* \otimes \mathcal{V}^*$	$(0,3)$-tensor

$$\binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} = 1 + 3 + 3 + 1 = 8$$

Chapter 2

tensor calculus

univariable derivative

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}f(x)}{\mathrm{d}x} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

add rule

$$\frac{\mathrm{d}(f+g)}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(f+g) = \frac{\mathrm{d}f}{\mathrm{d}x} + \frac{\mathrm{d}g}{\mathrm{d}x}$$

$$\frac{\mathrm{d}(f(x)+g(x))}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(f(x)+g(x)) = \frac{\mathrm{d}f(x)}{\mathrm{d}x} + \frac{\mathrm{d}g(x)}{\mathrm{d}x}$$

chain rule

$$\frac{\mathrm{d}f(g)}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}g}\frac{\mathrm{d}g}{\mathrm{d}x}$$

$$\frac{\mathrm{d}f(g(x))}{\mathrm{d}x} = \frac{\mathrm{d}f(g(x))}{\mathrm{d}g(x)}\frac{\mathrm{d}g(x)}{\mathrm{d}x}$$

univariate integral

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$$

multivariable derivative

$$\frac{\partial f}{\partial x} = \frac{\partial f(x,y)}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

add rule

$$\frac{\partial(f+g)}{\partial x} = \frac{\partial}{\partial x}(f+g) = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial x}$$

$$\frac{\partial(f(x,y)+g(x,y))}{\partial x} = \frac{\partial}{\partial x}(f(x,y)+g(x,y)) = \frac{\partial f(x,y)}{\partial x} + \frac{\partial g(x,y)}{\partial x}$$

chain rule

$$f(x,y) = f(x(t), y(t)) = f(t)$$

$$\frac{\mathrm{d}f(t)}{\mathrm{d}t} = \frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}$$

EpsilonDelta˜Ambiguity With Partial ∂ Notation, and How to Resolve It
$z = z(x,y)$:

$$\frac{\partial z}{\partial x} = \lim_{\Delta x\to 0} \frac{z(x+\Delta x, y) - z(x,y)}{\Delta x}$$
$$\frac{\partial z}{\partial y} = \lim_{\Delta y\to 0} \frac{z(x, y+\Delta y) - z(x,y)}{\Delta y}$$

$z = z(x(t), y(t))$:

$$\begin{aligned}
\frac{\mathrm{d}z}{\mathrm{d}t} &= \lim_{\Delta t \to 0} \frac{z(t+\Delta t) - z(t)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{z(x(t+\Delta t), y(t+\Delta t)) - z(x(t), y(t))}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{z(x(t+\Delta t), y(t+\Delta t)) - z(x(t), y(t+\Delta t)) + z(x(t), y(t+\Delta t)) - z(x(t), y(t))}{\Delta t} \\
&= \lim_{\Delta t \to 0} \left[\frac{z(x(t+\Delta t), y(t+\Delta t)) - z(x(t), y(t+\Delta t))}{\Delta t} + \frac{z(x(t), y(t+\Delta t)) - z(x(t), y(t))}{\Delta t} \right] \\
&= \lim_{\Delta t \to 0} \left[\frac{z(x(t+\Delta t), y(t+\Delta t)) - z(x(t), y(t+\Delta t))}{x(t+\Delta t) - x(t)} \frac{x(t+\Delta t) - x(t)}{\Delta t} + \right. \\
&\qquad \left. \frac{z(x(t), y(t+\Delta t)) - z(x(t), y(t))}{y(t+\Delta t) - y(t)} \frac{y(t+\Delta t) - y(t)}{\Delta t} \right] \\
&\approx \lim_{\Delta t \to 0} \left[\frac{z(x+\Delta x, y(t+\Delta t)) - z(x, y(t+\Delta t))}{\Delta x} \frac{\Delta x}{\Delta t} + \right. \\
&\qquad \left. \frac{z(x(t), y+\Delta y) - z(x(t), y)}{\Delta y} \frac{\Delta y}{\Delta t} \right] \\
&\approx \lim_{\Delta t \to 0} \left[\frac{z(x+\Delta x, y) - z(x, y)}{\Delta x} \frac{\Delta x}{\Delta t} + \frac{z(x, y+\Delta y) - z(x, y)}{\Delta y} \frac{\Delta y}{\Delta t} \right] \\
&= \lim_{\Delta t \to 0} \frac{z(x+\Delta x, y) - z(x, y)}{\Delta x} \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} + \lim_{\Delta t \to 0} \frac{z(x, y+\Delta y) - z(x, y)}{\Delta y} \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} \\
&= \frac{\partial z}{\partial x} \frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y} \frac{\mathrm{d}y}{\mathrm{d}t} \\
\frac{\mathrm{d}z}{\mathrm{d}t} &= \frac{\partial z}{\partial x} \frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y} \frac{\mathrm{d}y}{\mathrm{d}t}
\end{aligned}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x} \frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y} \frac{\mathrm{d}y}{\mathrm{d}t}$$

$z = z(x(s,t), y(s,t))$:

$$\begin{aligned}
\frac{\partial z}{\partial t} &= \lim_{\Delta t \to 0} \frac{z(s, t+\Delta t) - z(s, t)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \left[\frac{z(x(s, t+\Delta t), y(s, t+\Delta t)) - z(x(s,t), y(s, t+\Delta t))}{x(s, t+\Delta t) - x(s,t)} \frac{x(s, t+\Delta t) - x(s,t)}{\Delta t} + \right. \\
&\qquad \left. \frac{z(x(s,t), y(s, t+\Delta t)) - z(x(s,t), y(s,t))}{y(s, t+\Delta t) - y(s,t)} \frac{y(s, t+\Delta t) - y(s,t)}{\Delta t} \right] \\
&\approx \lim_{\Delta t \to 0} \frac{z(x+\Delta x, y) - z(x, y)}{\Delta x} \lim_{\Delta t \to 0} \frac{x(s, t+\Delta t) - x(s,t)}{\Delta t} + \\
&\qquad \lim_{\Delta t \to 0} \frac{z(x, y+\Delta y) - z(x, y)}{\Delta y} \lim_{\Delta t \to 0} \frac{y(s, t+\Delta t) - y(s,t)}{\Delta t} \\
&= \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} \\
\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}
\end{aligned}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}$$

$u = u(x(t), y(t), t)$:

$$\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}t} &= \lim_{\Delta t\to 0} \frac{u(t+\Delta t) - u(t)}{\Delta t} \\
&= \lim_{\Delta t\to 0} \frac{u(x(t+\Delta t), y(t+\Delta t), t+\Delta t) - u(x(t), y(t), t)}{\Delta t} \\
&= \lim_{\Delta t\to 0} \Bigg[\frac{u(x(t+\Delta t), y(t+\Delta t), t+\Delta t) - u(x(t), y(t+\Delta t), t+\Delta t)}{\Delta t} + \\
&\quad \frac{u(x(t), y(t+\Delta t), t+\Delta t) - u(x(t), y(t), t+\Delta t)}{\Delta t} + \\
&\quad \frac{u(x(t), y(t), t+\Delta t) - u(x(t), y(t), t)}{\Delta t} \Bigg] \\
&= \lim_{\Delta t\to 0} \Bigg[\frac{u(x(t+\Delta t), y(t+\Delta t), t+\Delta t) - u(x(t), y(t+\Delta t), t+\Delta t)}{x(t+\Delta t) - x(t)} \frac{x(t+\Delta t) - x(t)}{\Delta t} + \\
&\quad \frac{u(x(t), y(t+\Delta t), t+\Delta t) - u(x(t), y(t), t+\Delta t)}{y(t+\Delta t) - y(t)} \frac{y(t+\Delta t) - y(t)}{\Delta t} + \\
&\quad \frac{u(x(t), y(t), t+\Delta t) - u(x(t), y(t), t)}{t+\Delta t - t} \frac{t+\Delta t - t}{\Delta t} \Bigg] \\
&\approx \lim_{\Delta t\to 0} \Bigg[\frac{u(x+\Delta x, y, t) - u(x, y, t)}{\Delta x} \frac{\Delta x}{\Delta t} + \\
&\quad \frac{u(x, y+\Delta y, t) - u(x, y, t)}{\Delta y} \frac{\Delta y}{\Delta t} + \\
&\quad \frac{u(x, y, t+\Delta t) - u(x, y, t)}{\Delta t} \frac{\Delta t}{\Delta t} \Bigg] \\
&= \frac{\partial u}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial u}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial u}{\partial t}\frac{\mathrm{d}t}{\mathrm{d}t} = \frac{\partial u}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial u}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial u}{\partial t}\cdot 1 = \frac{\partial u}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial u}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial u}{\partial t} \\
&= \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial u}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} \\
\frac{\mathrm{d}u}{\mathrm{d}t} &= \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial u}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\partial u}{\partial t} + \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)\cdot\left(\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}\right) = \frac{\partial u}{\partial t} + \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) u \cdot \frac{\mathrm{d}}{\mathrm{d}t}(x, y) \\
&= \frac{\partial u}{\partial t} + \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) u \cdot \frac{\mathrm{d}}{\mathrm{d}t}(x, y) = \frac{\partial u}{\partial t} + \vec{\nabla} u \cdot \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \frac{\partial u}{\partial t} + \vec{\nabla} u \cdot \vec{v}
\end{aligned}$$

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial u}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\partial u}{\partial t} + \vec{\nabla} u \cdot \vec{v}$$

$u = u(x(t), y(t), t, z)$: ambiguity

$$\frac{\partial u}{\partial t} = \begin{cases} \lim_{\Delta t\to 0} \dfrac{u(x(t), y(t), t+\Delta t, z) - u(t)}{\Delta t} \\ \lim_{\Delta t\to 0} \dfrac{u(x(t+\Delta t), y(t+\Delta t), t+\Delta t, z) - u(t)}{\Delta t} \end{cases} = \lim_{\Delta t\to 0} \frac{u(t+\Delta t) - u(t)}{\Delta t}$$

solution 1: functional approach: PDE(partial differential equation), QM(quantum mechanics)

solution 2: equational approach: TD(thermodynamics), equilibrium chemistry

solution 1: functional approach:

$u = u\left(x\left(t\right), y\left(t\right), t, z\right)$:

$$u = u\left(x\left(t\right), y\left(t\right), t, z\right) \Rightarrow \begin{cases} u = f\left(x, y, z, t\right) \\ x = g\left(t\right) \\ y = h\left(t\right) \end{cases}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f\left(x, y, z, t\right)}{\partial t}$$

$$u = u\left(x\left(t\right), y\left(t\right), t, z\right) \Rightarrow \begin{cases} u = u\left(x, y, z, t\right) \\ x = x\left(t\right) \\ y = y\left(t\right) \end{cases}$$

$$\frac{\partial u}{\partial t} = \begin{cases} \dfrac{\partial u\left(x, y, z, t\right)}{\partial t} & = \left(\dfrac{\partial u}{\partial t}\right)_{x,y} = \left(u_t\right)_{x,y} \\ \dfrac{\partial u\left(x\left(t\right), y\left(t\right), t, z\right)}{\partial t} & = \dfrac{\partial u}{\partial t} \end{cases}$$

function:
parametric functions

$$x\left(t\right), y\left(t\right)$$

wave function

$$\psi\left(x, y, t\right)$$

equation:
ideal gas law

$$PV = nRT$$

$$\begin{aligned} PV &= nRT \\ \frac{\partial P}{\partial T} &= \\ \left(\frac{\partial P}{\partial T}\right)_{V,n} &= \\ \left(\frac{\partial P}{\partial T}\right)_{V/n} = \left(\frac{\partial P}{\partial T}\right)_{\frac{V}{n}} &= \end{aligned}$$

$x_1 + x_2 + x_3 = 1$: e.g. alloy metal (ternary alloy = 3 metals) phase diagram

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ x_1 + x_3 &= 1 - x_2 \\ x_1\left(1 + \frac{x_3}{x_1}\right) &= 1 - x_2 \\ x_1 &= \frac{1 - x_2}{1 + \frac{x_3}{x_1}} \\ \left(\frac{\partial x_1}{\partial x_2}\right)_{\frac{x_3}{x_1}} &= \frac{\partial}{\partial x_2}\frac{1 - x_2}{1 + \frac{x_3}{x_1}} = \frac{\frac{\partial}{\partial x_2}\left(1 - x_2\right)}{1 + \frac{x_3}{x_1}} = \frac{0 - \frac{\partial x_2}{\partial x_2}}{1 + \frac{x_3}{x_1}} = \frac{-1}{1 + \frac{x_3}{x_1}} \\ &= \frac{-1}{1 + \frac{x_3}{x_1}} = \frac{-1 \cdot x_1}{1 \cdot x_1 + \frac{x_3}{x_1} \cdot x_1} = \frac{-x_1}{x_1 + x_3} = \frac{-x_1}{1 - x_2} \qquad x_1 + x_3 = 1 - x_2 \end{aligned}$$

$u = u\left(x\left(t\right), y\left(t\right), t\right)$:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial u}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\partial u}{\partial t} + \vec{\nabla}u \cdot \vec{v}$$

$u = u\left(x\left(t\right), y\left(t\right), t, z\right)$:

$$\frac{\partial u}{\partial t} = \left(\frac{\partial u}{\partial t}\right)_{x,y} + \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$$

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial u}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\partial u}{\partial t} + \vec{\nabla} u \cdot \vec{v} \qquad u = u\left(x\left(t\right), y\left(t\right), t\right)$$

$$\frac{\partial u}{\partial t} = \left(\frac{\partial u}{\partial t}\right)_{x,y} + \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} \qquad u = u\left(x\left(t\right), y\left(t\right), t, z\right)$$

context-dependent notation of $\frac{\partial u}{\partial t}$

explicit partial derivative vs. complete partial derivative

$$\underbrace{\frac{\mathrm{d}u}{\mathrm{d}t}}_{\text{total derivative}} = \underbrace{\frac{\partial u}{\partial t}}_{\text{explicit partial}} + \frac{\partial u}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial u}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\partial u}{\partial t} + \vec{\nabla} u \cdot \vec{v} \qquad u = u\left(x\left(t\right), y\left(t\right), t\right)$$

$$\underbrace{\frac{\partial u}{\partial t}}_{\text{complete derivative}} = \left(\frac{\partial u}{\partial t}\right)_{x,y} + \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} \qquad u = u\left(x\left(t\right), y\left(t\right), t, z\right)$$

convective derivative vs. material derivative

$$\frac{\mathrm{D}u}{\mathrm{D}t} = \frac{\mathrm{d}u}{\mathrm{d}t} = \overbrace{\frac{\partial u}{\partial t} + \underbrace{\frac{\partial u}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial u}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}}_{\text{convective derivative}}}^{\text{material derivative}} = \frac{\partial u}{\partial t} + \vec{\nabla} u \cdot \vec{v}$$

This is why we can't have nice thing!