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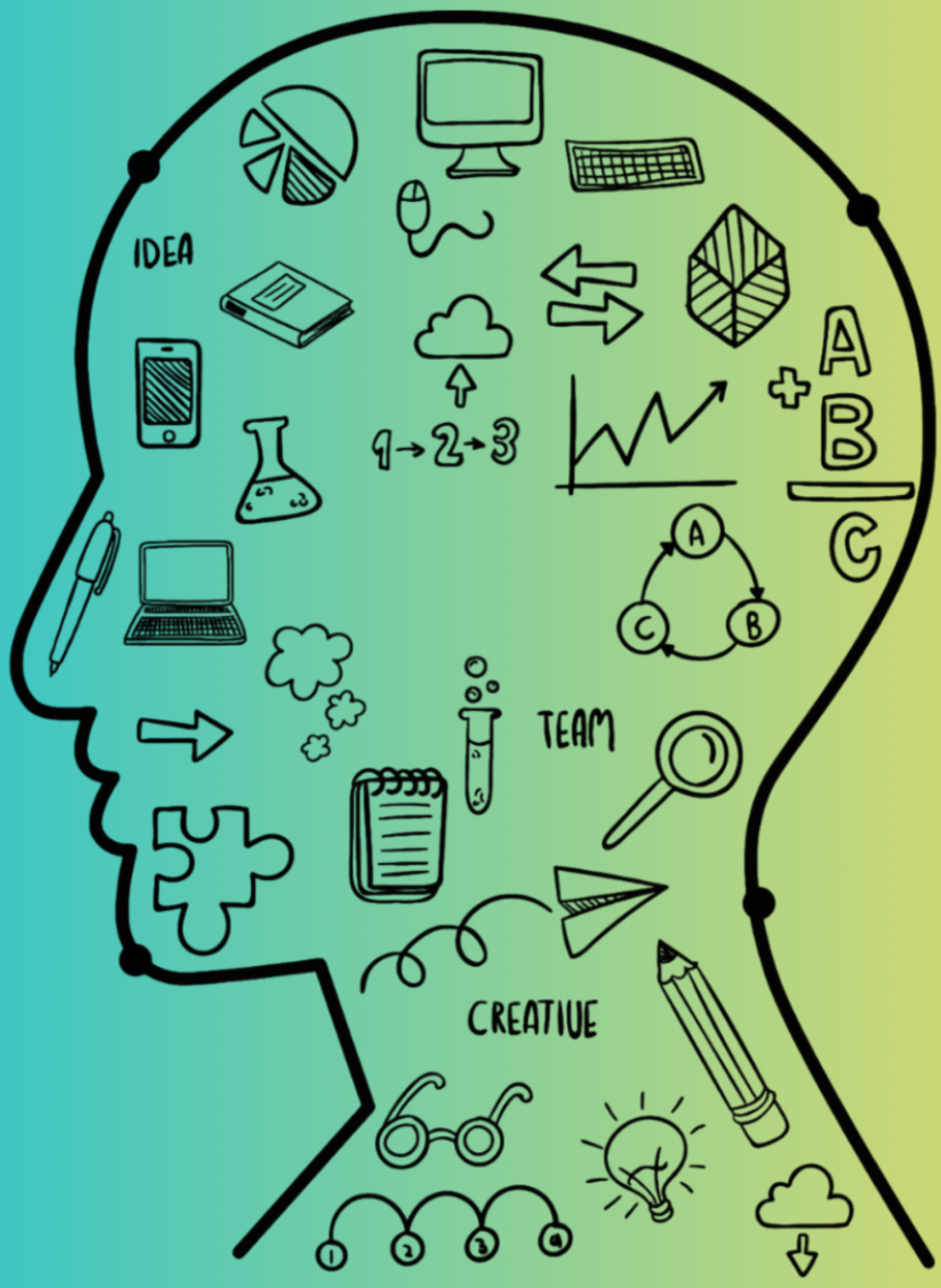
CALCULUS AND ITS APPLICATIONS

Mathematical Techniques



Written by:

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First Edition

Calculus and Its Applications

Mathematical Techniques

Bakti Siregar, M.Sc., CDS.

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In the evolving landscape of science, engineering, and technology, calculus remains a fundamental tool for understanding change, modeling complex systems, and solving real-world problems. From the classical challenges of motion and geometry to modern applications in data science, optimization, and engineering, calculus provides a unifying language that bridges theory and practice. By mastering its core concepts and techniques, students and practitioners can analyze dynamic processes, make informed decisions, and develop models that foster innovation and excellence across disciplines.

This book, Calculus and Its Applications: Mathematical Techniques, offers a structured and comprehensive introduction to calculus. Beginning with the foundations of real numbers and the essentials of functions, readers are gradually guided through special functions, limits, and the core principles of derivatives. Building on these fundamentals, the text explores both the applications of derivatives in optimization and modeling, as well as the theory and practice of indefinite integrals and their wide-ranging applications. The journey concludes with a discussion of transcendental functions, connecting classical concepts to advanced and contemporary challenges.

Beyond theory, the book emphasizes practical applications—showing how calculus underpins decision-making, system optimization, and problem-solving in diverse fields. Each chapter integrates concepts with examples that reflect both traditional mathematical problems and modern technological contexts.

Through this approach, readers will not only develop a strong understanding of the mathematical principles of calculus but also gain the skills to apply them effectively to real-world challenges—fulfilling the book's vision of connecting classical problems with modern challenges.

Preface

About the Writer

Bakti Siregar, M.Sc., CDS works as a Lecturer at the ITSB Data Science Program. He earned his Master's degree from the Department of Applied Mathematics at National Sun Yat Sen University, Taiwan. In addition to teaching, Bakti also works as a Freelance Data Scientist for leading companies such as JNE, Samora Group, Pertamina, and PT. Green City Traffic.

He has a strong enthusiasm for projects (and teaching) in the fields of Big Data Analytics, Machine Learning, Optimization, and Time Series Analysis, particularly in finance and investment. His core expertise lies in statistical programming languages such as R Studio and Python. He is also experienced in implementing database systems like MySQL/NoSQL for data management and is proficient in using Big Data tools such as Spark and Hadoop.

Some of his projects can be viewed here: Rpubs, Github, Website, and Kaggle

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Acknowledgments

Calculus plays a vital role in modeling, analyzing, and optimizing processes across science, engineering, and technology. This book introduces fundamental concepts and techniques in calculus, including:

- A solid foundation in real numbers, functions, and limits
- The ability to analyze and interpret data across engineering and scientific contexts
- A clear understanding of the role of derivatives and integrals in modeling and problem-solving
- Practical skills in applying numerical methods and calculus techniques to real-world challenges

This book is designed for beginners seeking to build a strong foundation in calculus while appreciating its concepts and diverse applications—from classical mathematical problems to modern scientific and engineering challenges. We value the active participation of readers, whose insights and questions enrich the learning journey. It is our hope that this material serves not only as an introduction to calculus but also as a practical guide for applying mathematical reasoning to contemporary problems.

Feedback & Suggestions

Your feedback is invaluable in enhancing the quality of this book. We warmly invite readers to share their thoughts on the content, organization, and clarity of the material. Suggestions for additional topics, extended explanations, or further real-world applications are highly encouraged.

With your support and contributions, our goal is to make this book a comprehensive and accessible resource on calculus and its applications—from classical problems to modern challenges. Thank you for your engagement and feedback.

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Chapter 1

Introduction to Calculus

Calculus (Mathematical Techniques) is a branch of mathematics that helps us understand how things **change** (such as population growth, temperature rise, or production rates) and how things **move** (like a car on the road, an object falling, or material flowing in a system). It provides useful tools to **solve problems**, make predictions, and create models in science, engineering, and daily life. In **Mining** whether in oil, coal, gold, copper, or other mineral resources—calculus is applied to estimate reserves, calculate safe and efficient extraction methods, model slope stability, predict groundwater flow, monitor production performance, and optimize operational aspects such as cost efficiency, revenue generation, and workforce allocation. In **Metallurgy,** it is used to analyze heating and cooling processes, **optimize** smelting operations, study chemical reaction rates, design stronger alloys, and predict corrosion over time. Overall, calculus is not just about numbers but a practical tool that supports efficiency, safety, and **sustainability** in industries such as mining, metallurgy, and many other fields.

1.1 Overview

The Figure 1.1 presents a visual overview of the chapter, highlighting the structure of key topics and their interconnections. It provides readers with a clear guide to navigate the material and understand ho w concepts link to applications.

This chapter introduces the fundamental building blocks of calculus, including real numbers, functions, limits, derivatives, integrals, and transcendental functions. Each concept is connected to practical applications to illustrate how calculus underpins real-world problem-solving.

1.2 Applied Calculus

Applied Calculus (Figure 1.2) is a branch of calculus that focuses on the practical application of limits, derivatives, integrals, and special functions in solving real-world problems. Rather than being studied purely as a theoretical subject, calculus is presented here as an analytical tool to understand change, model phenomena, and support decision-making across multiple disciplines.

The following mindmap illustrates a 5W+1H framework that explains:

Figure 1.1: Mind Map of Introduction to Calculus

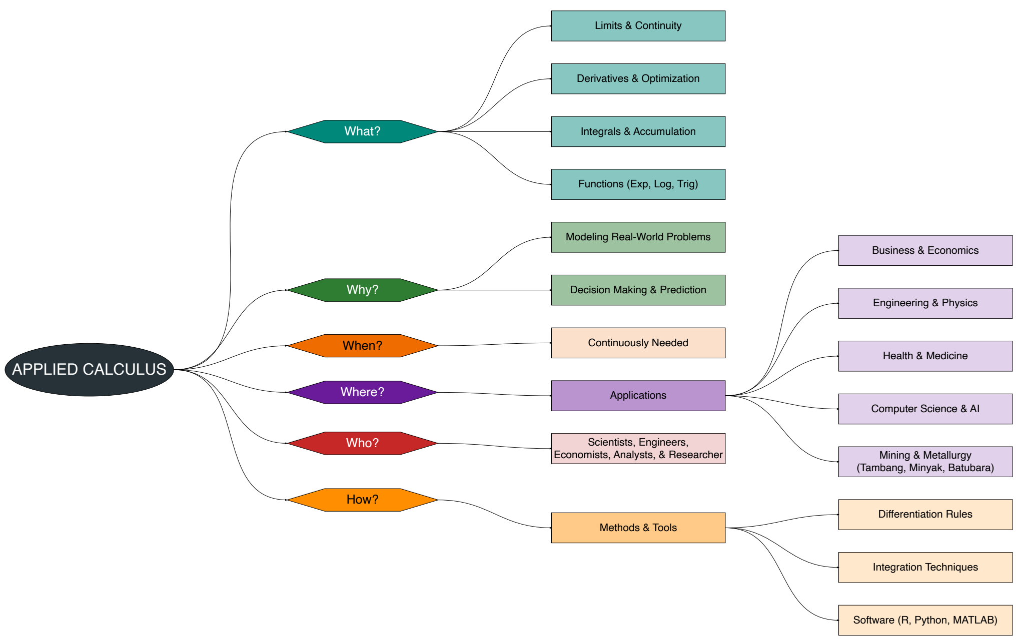

Figure 1.2: Detailed 5W+1H for Applied Calculus

- **What:** The core concepts of Applied Calculus, including limits, derivatives, integrals, and special functions.
- **Why:** Its importance in modeling real-world problems, prediction, and decision-making.
- **When:** Applied Calculus has always been important in the past, remains essential today, and will continue to be needed in the future for science, technology, and real-world applications.
- **Where:** The areas of application, ranging from business and economics, engineering and physics, health and medicine, computer science and AI, to mining engineering and metallurgy.
- **Who:** The practitioners who apply Applied Calculus, such as scientists, engineers, economists, and data analysts.
- **How:** The methods and tools used, both analytical techniques and modern computational software.

This structure helps learners understand the role of calculus as an applied, relevant, and essential discipline for addressing challenges across diverse fields.

Chapter 2

Real Numbers

Understanding **Real Numbers ($\mathbb{R}$)** is the first step in exploring the world of **real analysis**. These numbers serve as the essential building blocks for calculus, algebra, numerical modeling, and various applied sciences. They provide a framework for representing quantities, measuring change, and describing continuous processes in both mathematics and real-world applications [1]–[3].

To help navigate the key aspects of real numbers, the Figure 2.1 offers a **5W+1H mind map**. This visualization guides learners through the **What**—definitions and subsets; the **Why**—their importance and significance; the **When**—historical discoveries and formalization; the **Where**—applications in science, engineering, economics, and daily life; the **Who**—mathematicians and everyday users; and the **How**—representation on the number line, decimal forms, and intervals. By following this map, one can see not just the numbers themselves, but their role and relevance across disciplines.

The following Table 2.1 presents a structured summary of the **5W+1H questions** related to Real Numbers, based on the Figure 2.1 mind map. It organizes the material into categories—**What, Why, When, Where, Who, How**—to guide learners in understanding the definitions, subsets, properties, number line representation, and applications of real numbers in science, engineering, economics, and daily life.

2.1 Definition

The real numbers ($\mathbb{R}$) are the set of numbers that include both rational numbers (fractions of integers) and irrational numbers (numbers that cannot be expressed as fractions). They can be represented on the number line, which extends infinitely in both positive and negative directions [1], [2].

Formally:

$$\mathbb{R} = \{x \mid x \text{ corresponds to a point on the number line}\}.$$

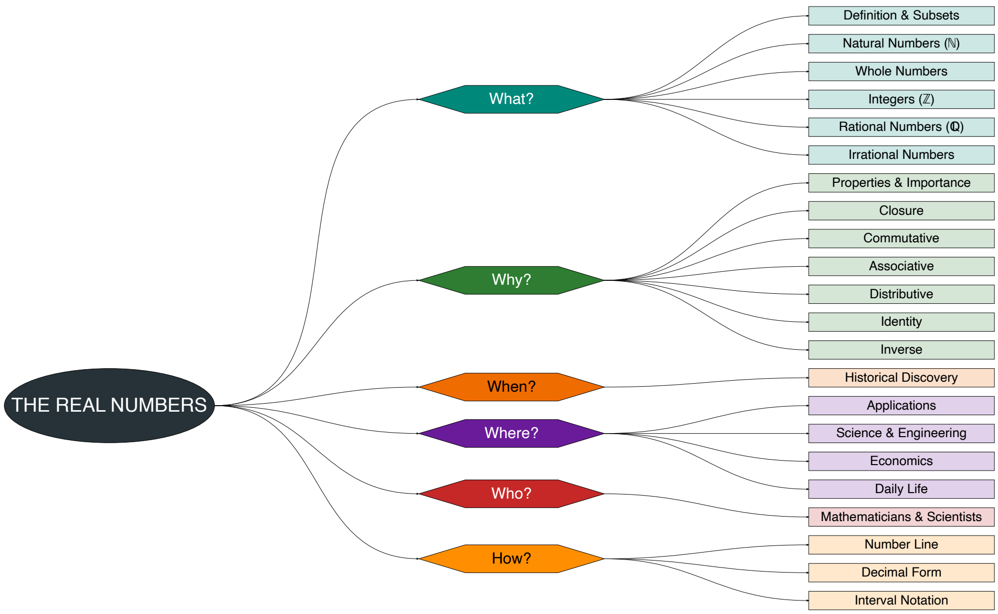

Figure 2.1: Real Numbers with 5W+1H Notes

Table 2.1: Understanding the Properties of Real Numbers

Property	**Description**	**Example / Application**
Closure	Operations on real numbers (add, subtract, multiply, divide) always produce another real number.	$a+b$, $a-b$, $a \cdot b$, a/b (if $b \neq 0$)
Order	Real numbers can be compared and arranged from smallest to largest.	$a < b$ → comparing magnitudes, sorting data
Density	Between any two real numbers, there is always another real number — useful for interpolation and fine measurement.	$\exists c : a < c < b$ → interpolation or precision scale
Absolute Value	Measures distance from zero regardless of sign.	$\lvert a \rvert$ → magnitude of deviation or change
Scientific Measurement	Represents measurable quantities in science, mining, and metallurgy (mass, distance, temperature, ore grades).	Mass = 1.2 × 10□ kg, Temperature = 350°C, Depth = 500 m
Engineering Constants	Physical constants used in formulas and calculations (e.g., gravity, speed of light).	$g = 9.8\,\text{m/s}^2$, $c = 3 \times 10^8$ m/s
Negative Numbers	Represents losses, deficits, or values below a reference point (e.g., depth below sea level, financial loss).	Depth = −350 m, Balance = −$50, Temperature = −5°C
Fractions / Decimals	Represents portions or decimal values (e.g., ore grade percentage, sample concentration).	Ore grade = 2.5%, Concentration = 0.025, Tax = 7.5%
Irrational Numbers	Non-repeating, non-terminating decimals used in precise measurements (e.g., π, √2).	$\pi = 3.14159...$, $\sqrt{2} = 1.4142...$, $e = 2.718...$

2.2 Subsets

Real numbers ($\mathbb{R}$) consist of several subsets, each with distinct properties and applications. Understanding these subsets is fundamental in mathematics, physics, and engineering.

2.2.1 Natural Numbers ($\mathbb{N}$)

Natural numbers are the set of positive counting numbers used for enumerating objects. Formally, the set of natural numbers is written as

$$\mathbb{N} = \{1, 2, 3, 4, \ldots\}.$$

Natural numbers have several important properties. They are always positive, and they are closed under addition and multiplication. However, they are not closed under subtraction or division; for example, $2 - 3 \notin \mathbb{N}$. Some examples of natural numbers include $1, 2, 3, 10, 100$, and so on. These numbers are widely used in everyday life and in mathematics for counting discrete objects, numbering sequences, and performing basic arithmetic operations.

2.2.2 Whole Numbers

Whole numbers extend natural numbers by including zero. Formally, the set of whole numbers is written as

$$\text{Whole Numbers} = \{0, 1, 2, 3, \ldots\}.$$

Whole numbers have several important properties. They are non-negative and are closed under addition and multiplication. However, they are not closed under subtraction; for example, $0 - 1 \notin$ Whole Numbers. Some examples of whole numbers include $0, 1, 2, 50, 1000$, and so on. Whole numbers are widely used in numbering positions, indexing in programming, and counting objects when zero is included.

2.2.3 Integers ($\mathbb{Z}$)

Integers include all whole numbers and their negative counterparts. Formally, the set of integers is written as

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}.$$

Integers have several important properties. They are closed under addition, subtraction, and multiplication, but they are not closed under division; for example, $1/2 \notin \mathbb{Z}$. Some examples of integers include $-10, -1, 0, 3, 15$, and so on. Integers are widely used for representing gains and losses, elevations, temperatures, and positions relative to a reference point.

2.2.4 Rational Numbers ($\mathbb{Q}$)

Rational numbers are numbers that can be expressed as a fraction $\frac{p}{q}$ where $p, q \in \mathbb{Z}$ and $q \neq 0$. Formally, the set of rational numbers is written as

$$\mathbb{Q} = \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0 \right\}.$$

Rational numbers have several important properties. They can be positive, negative, or zero, and they are closed under addition, subtraction, multiplication, and division (except division by zero). They can also be represented as terminating or repeating decimals. Some examples of rational numbers include $\frac{1}{2}, -\frac{7}{3}, 0.75$, and $0.333\ldots$. Rational numbers are widely used in fractions for measurements, probabilities, ratios, and proportional relationships.

2.2.5 Irrational Numbers

Irrational numbers cannot be expressed as a fraction $\frac{p}{q}$ with integers p and q, and their decimal expansions are non-terminating and non-repeating. They can be positive or negative and are generally closed under addition, subtraction, multiplication, and sometimes division, but they cannot be represented exactly as a fraction. Examples of irrational numbers include π, e, $\sqrt{2}$, $\sqrt{3}$, and $\ln 2$. These numbers are widely used in geometry, such as π for circles, in calculus, in physical constants, and in modeling exponential growth or decay.

2.3 Properties

The set of real numbers ($\mathbb{R}$) follows several fundamental rules that govern arithmetic operations, essential in algebra, calculus, and applied mathematics.

2.3.1 Closure

$\mathbb{R}$ is **closed under addition and multiplication**, meaning the sum or product of any two members is still a real number. Division by zero is the only exception.

2.3.2 Commutative

Addition and multiplication are **commutative**, so the order of numbers does not affect the result:

$$a + b = b + a \quad \text{and} \quad a \cdot b = b \cdot a.$$

2.3.3 Associative

Grouping of numbers does not change the outcome, reflecting the **associative rule** for both operations:

$$(a + b) + c = a + (b + c) \quad \text{and} \quad (a \cdot b) \cdot c = a \cdot (b \cdot c).$$

2.3.4 Distributive

Multiplication distributes over addition, meaning multiplying a number by a sum equals multiplying each term individually and then adding:

$$a \cdot (b + c) = a \cdot b + a \cdot c.$$

2.3.5 Identity

There exist **identities** for addition and multiplication. Adding 0 or multiplying by 1 leaves any number unchanged:

$$a + 0 = a \quad \text{and} \quad a \cdot 1 = a.$$

2.3.6 Inverse

Every number has **additive and multiplicative inverses** (except zero for multiplication). The additive inverse $-a$ satisfies $a + (-a) = 0$, and the reciprocal $\frac{1}{a}$ satisfies $a \cdot \frac{1}{a} = 1$.

2.4 Representation

Figure Figure 2.2 illustrates a number line, a visual tool representing real numbers in order, which helps to clearly understand their relative positions and relationships.

The number line is a fundamental visual tool in mathematics that allows us to represent real numbers in order. It provides a clear way to understand the relative positions of numbers, including zero as the central reference point, positive numbers to the right, and negative numbers to the left. Rational numbers can be located precisely on the line, while irrational numbers occupy approximate positions between integers, filling in the gaps and illustrating the density of real numbers. The key concepts summarized in Table Table 2.2 highlight the main categories and their properties, helping to organize the understanding of real numbers on the number line.

2.5 Summary

Real numbers Table 2.3 are the **foundation of all calculus concepts**, forming the set of numbers that includes integers, fractions, and decimals. They are essential for performing calculations, defining functions, and understanding limits, derivatives, and integrals. A solid understanding of real numbers allows us to work with continuous quantities, measure physical phenomena, and apply mathematical reasoning in real-world contexts. The key properties, descriptions, and typical applications of real numbers are summarized in Table 2.4.

2.6 Youtube ~ Real Numbers

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Watch here: Click here to watch the video

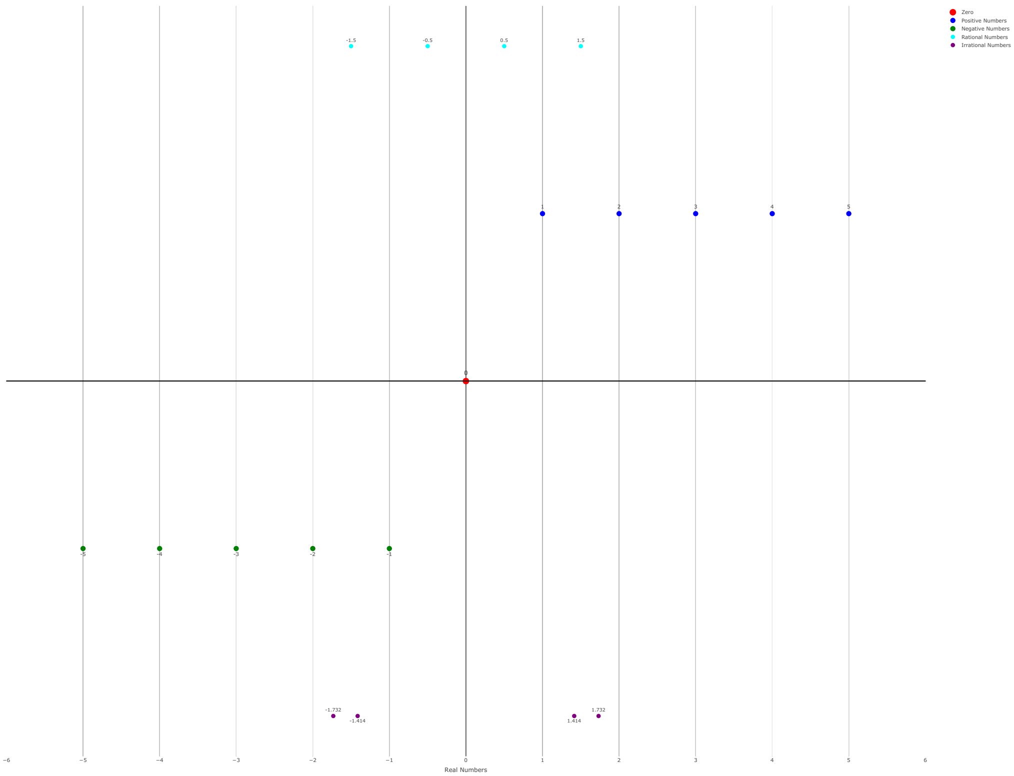

Figure 2.2: Representation on Number Line

Table 2.2: Key Concepts of Number Line

Concept	Description	Notes
Zero as Center	Central reference point separating positive and negative numbers.	Acts as reference for measuring distan direction (Figure 2.2)
Positive Numbers	Numbers greater than zero placed to the right of the origin; includes natural, whole, and positive fractions or decimals.	Magnitude increases to the right of zer
Negative Numbers	Numbers less than zero placed to the left of the origin; represents deficits, losses, or positions below reference.	Includes negative fractions and decima
Rational Numbers	Numbers expressible as fractions or terminating/repeating decimals; located exactly on the number line.	Each fraction corresponds to a precise between integers
Irrational Numbers	Numbers not exactly expressible as fractions; approximate positions between integers filling in the gaps.	Examples: π, $\sqrt{2}$, e; shows density of r numbers

Table 2.3: Understanding the Properties of Real Numbers

Property	Description	Example
Closure	Operations on real numbers (add, subtract, multiply, divide) always produce another real number.	$a + b$, $a - b$, $a \cdot b$, a/b (if $b \neq 0$)
Order	Real numbers can be compared and arranged from smallest to largest.	$a < b$ → comparing magnitudes, sorting data
Density	Between any two real numbers, there is always another real number — useful for interpolation and fine measurement.	$\exists c : a < c < b$ → interpolation, fine measurement
Absolute Value	Measures distance from zero regardless of sign.	$\|a\|$ → distance from zero, magnitude of change
Scientific Measurement	Represents measurable quantities in science, mining, and metallurgy (mass, distance, temperature, ore grades).	Coal reserve = 1.2M tons, Temperature = 350°C, Depth = 500 m
Engineering Constants	Physical constants used in formulas and calculations (e.g., gravity, speed of light).	$g = 9.8$ m/s^2, $c = 3 \times 10^8$ m/s
Negative Numbers	Represents losses, deficits, or values below a reference point (e.g., debt, depth below sea level, temperature below zero).	Bank balance = −$50, Depth = −350 m, Temperature = −5°C
Fractions / Decimals	Used for precise measurements, proportions, percentages, ore grades, or financial fractions.	Ore grade = 2.5%, Cooking 250 g flour, Tax = 7.5%
Irrational Numbers	Numbers like π, √2, or e appear in formulas, geometry, and scientific modeling.	Volume of cylinder = π·r²h, Diagonal of square = √2·a, Continuous growth = e^rt

2.7 Applied

Real numbers play a crucial role in many fields because they can represent continuous quantities, perform precise measurements, and quantify relationships. Table Table 2.4 summarizes their main applications in science and engineering, economics, and everyday life.

References

Table 2.4: Applications of Real Numbers (Positive, Negative, Zero, Fractions, Irrational)

Domain	**Description**	**Examples**
Science & Engineering	Real numbers (positive, negative, zero, fractions, and irrationals) represent continuous measurements like distance, mass, temperature, velocity, and energy. Negative values describe loss (e.g., temperature below zero), fractions handle precise scaling, and irrationals (like π, √2) appear in formulas and geometry.	Positive: 12.5 m, Zerc (direction), Fraction:
Economics & Business	In economics and business, real numbers express prices, costs, revenues, profits, debts, and interest rates. Negative numbers show losses or debts, fractions are used in taxation or discount rates, and irrationals sometimes appear in financial models and growth rates.	Price = $25.50, Profit Growth = e^rt
Daily Life	In daily life, real numbers appear in money, weights, volumes, percentages, and time. Negative values are seen in temperatures (−5 °C), balances (−$50), or altitude below sea level, while fractions and decimals are used in cooking or shopping.	−5 °C, Bank balance = Meeting at 14:30, Dis
Mining & Metallurgy	In mining and metallurgy, real numbers quantify mineral reserves, drilling depths, extraction rates, chemical concentrations, heat levels, costs, and revenues. Negative values reflect losses or deficits, zero indicates balance or cut-off limits, fractions express ore grades, and irrationals (like π in volume calculations) support precise modeling.	Coal reserve = 1.2M t 2.5%, Smelting T = 1

Chapter 3

Essencials of Functions

Understanding **Functions** is fundamental in mathematics, as they describe the relationship between quantities and form the backbone of calculus, algebra, numerical modeling, and applied sciences. Functions allow us to model change, describe systems, and solve real-world problems [1]–[3].

The **Figure** 3.1 shows a **5W+1H mind map** of functions.
It helps learners understand:

- **What** → definition, domain, range, and types of functions.
- **Why** → why functions are important and their applications in math and science.
- **When** → when functions are used, especially in problem-solving and research.
- **Where** → areas of application such as engineering, physics, economics, and daily life.
- **Who** → mathematicians and practitioners who use functions.
- **How** → ways to represent functions using equations, tables, graphs, and intervals.

Functions are one of the core concepts in mathematics, playing a central role in **modeling, analysis, and real-world applications**. Using the **5W+1H framework (What, Why, When, Where, Who, How)**, we can explore functions from multiple perspectives: their definition, importance, historical development, fields of application, key contributors, and different forms of representation.

Table Table 3.1 summarizes the key questions and provides illustrative examples of functions along with their interpretations for each 5W+1H category.

3.1 Definition

Imagine a Figure 3.2: You put in fruits (for example, 1 apple, 2 apples, or 3 apples). The machine processes the fruits. It always produces a certain amount of apple juice depending on how many

Table 3.1: 5W+1H Questions for Functions

	Description	Example_Function	Example_Output
What?			
What?	What is a function?	$f : X \to Y$	Each input → exactly one output
What?	What are the domain and range?	$f(x) = x^2$, Domain = $\mathbb{R}$, Range = $[0, \infty)$	Domain all inputs; Range all outp
What?	What types of functions exist?	Linear, Quadratic, Polynomial, Exponential, Trigonometric	Example: $f(x) = 2x + 1$, $f(x) = x^2 - 3$
What?	What are the key properties of functions?	Injective, Surjective, Bijective, Continuous, Monotone	e.g. $f(x) = x^2$ is not injective on
Why?			
Why?	Why are functions important in mathematics?	Modeling $y = f(x)$ relationships	Predict outcomes, solve equations
Why?	Why do we need to understand function properties?	Analyzing $f(x)$ before applying to problems	Correct manipulation of $f(x)$
When?			
When?	When was the function concept formalized?	17th century (Leibniz, Euler)	Formalized in 1600s
When?	When are functions applied in real-life problems?	Finance: $A(t) = P(1 + r)^t$; Physics: $s(t) = v_0 t + \frac{1}{2}at^2$	Applications in simulations and modeling
Where?			
Where?	Where are functions used in science and engineering?	Ohm's law: $V = IR$, Newton's law: $F = ma$	Used in circuits, mechanics, chemistry
Where?	Where can functions be observed in economics and daily life?	Population growth $P(t) = P_0 e^{rt}$	Used in demand curves, budgetin
Who?			
Who?	Who were key mathematicians in developing function theory?	Euler, Leibniz, Dirichlet	Pioneers in function theory
Who?	Who uses functions in practical applications?	Scientists, engineers, economists	Real-world users across disciplin
How?			
How?	How are functions represented using equations?	$f(x) = x^2$, $f(x) = \sin x$	Symbolic form representation
How?	How are functions represented using tables?	Tabular form: $(x, f(x))$ pairs	Input-output lookup
How?	How are functions represented using graphs or intervals?	Graph of $f(x)$, interval $[a, b]$	Visual/geometric representation

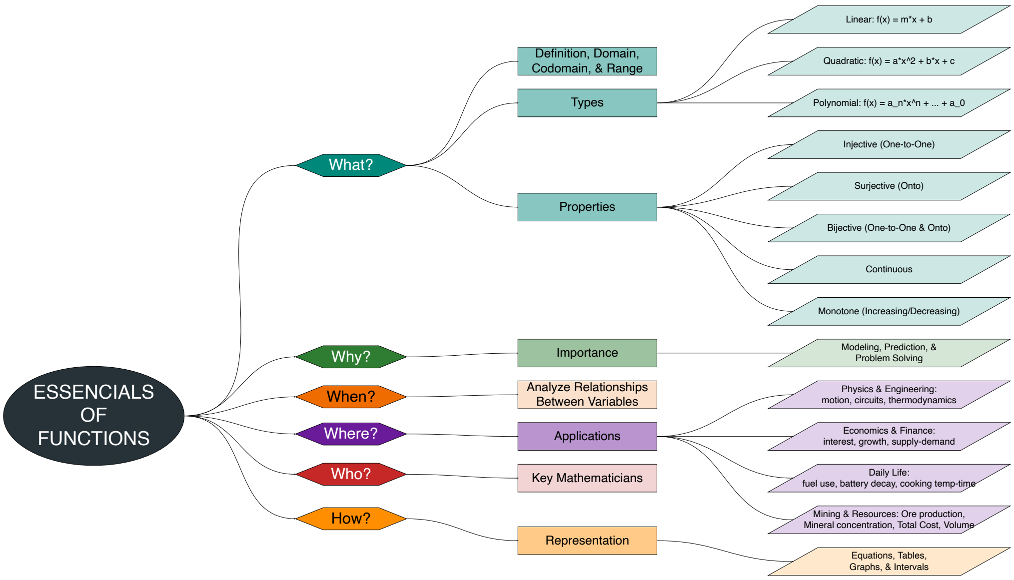

Figure 3.1: Detailed 5W+1H for Functions

apples you put in.

Figure 3.2: Analogies of Juice Machine to Understand About Function

The rule is clear: **each input has exactly one output**.

- If you put in 2 apples → you get 2 glasses of juice.
- If you put in 3 apples → you get 3 glasses of juice.

This is very similar to the concept of a **single-variable function** in mathematics. Assume a Linear Function (simple juice machine) as;

$$f(x) = 2x$$

Meaning: for every x (number of apples), the output is twice the input. Therefore if the Input: $x = 3 \rightarrow$ Output: $f(3) = 6$.

In General, A function f from set X to set Y is a rule that assigns **exactly one element of** Y to each element of X. Formally:

$$f : X \to Y \quad \text{such that } \forall x \in X, \exists! y \in Y \text{ with } y = f(x)$$

- **Domain**: the set of all inputs X
- **Range**: the set of all outputs Y

This definition ensures that every input has **one and only one output**, which distinguishes functions from more general relations. For example, consider $f(x) = x^2$ with domain $\mathbb{R}$. Each real number x is mapped to a single nonnegative real number $y = x^2$. In this case, the domain is $\mathbb{R}$ and the range is $\mathbb{R}_{\geq 0}$.

3.2 Types of Functions

Functions are fundamental tools in mathematics that describe the relationship between two quantities, typically denoted as an input x and an output $f(x)$. Each type of function has its own characteristics, shape, and application in real-world problems. Understanding these different types of functions is crucial not only in pure mathematics but also in various applied fields such as engineering, economics, physics, and mining engineering, where they are used to model growth, decay, oscillations, and relationships between variables.

Broadly, functions can be categorized into several groups, such as algebraic functions (linear, quadratic, polynomial), transcendental functions (exponential and logarithmic), and trigonometric functions (sine, cosine, tangent).

3.2.1 Algebraic

Algebraic functions (Figure Figure 3.3) are functions that can be expressed using a finite number of algebraic operations such as addition, subtraction, multiplication, division, and raising to a power. These functions form the foundation of many mathematical models and are widely applied in real-life problem solving.

The main types of algebraic functions include:

- **Linear Function**: $f(x) = mx + b$, which produce straight-line graphs and represent constant rates of change.
- **Quadratic Function**: $f(x) = ax^2 + bx + c$, which generate parabolic curves and are often used to model acceleration, projectile motion, or optimization problems.
- **Polynomial Function**: $f(x) = a_n x^n + \cdots + a_0$, which extend the idea of linear and quadratic functions to higher degrees, allowing the modeling of more complex relationships.

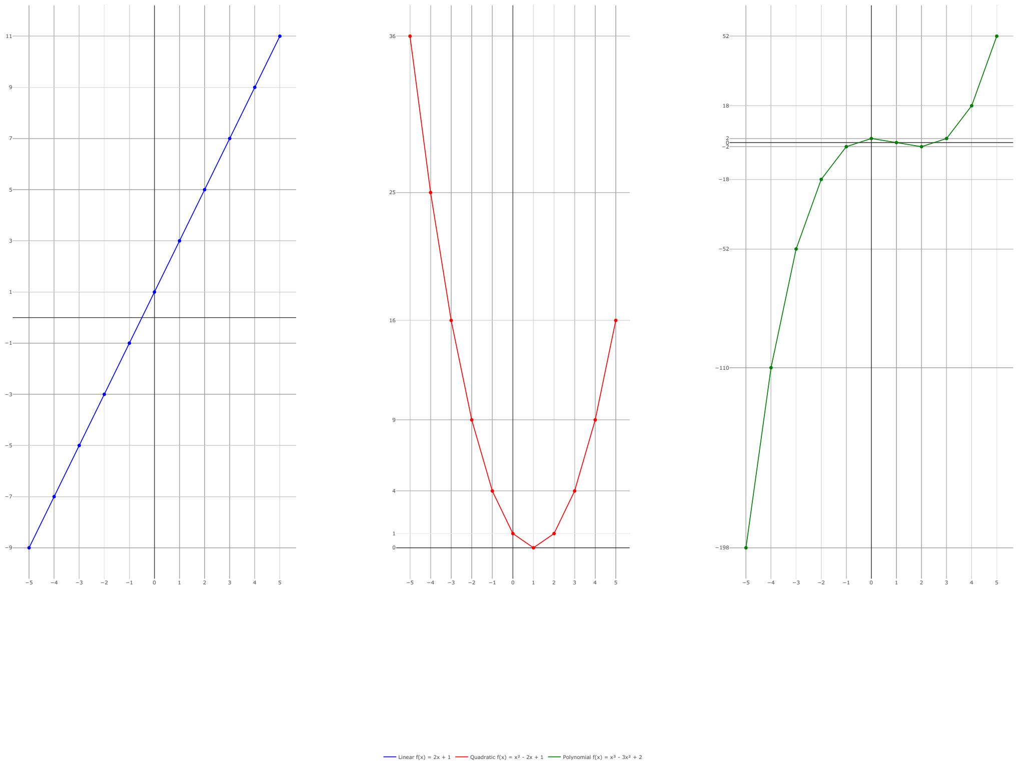

Figure 3.3: Algebraic Functions: Linear, Quadratic, and Polynomial (Side by Side)

3.2.2 Transcendental Functions

Transcendental functions (Figure Figure 3.4) are functions that cannot be expressed as finite combinations of algebraic operations. Unlike algebraic functions, they involve processes such as infinite series, exponentiation, and logarithms. These functions play a vital role in describing natural growth, decay, and scaling phenomena.

- **Exponential Function**: $f(x) = a^x$ are used to model rapid growth or decay, such as in population dynamics, radioactive decay, and compound interest.
- **Logarithmic Function**: $f(x) = \log_a x$ serve as the inverse of exponentials, commonly applied in measuring relative change, sound intensity (decibels), pH in chemistry, and data compression in computer science.

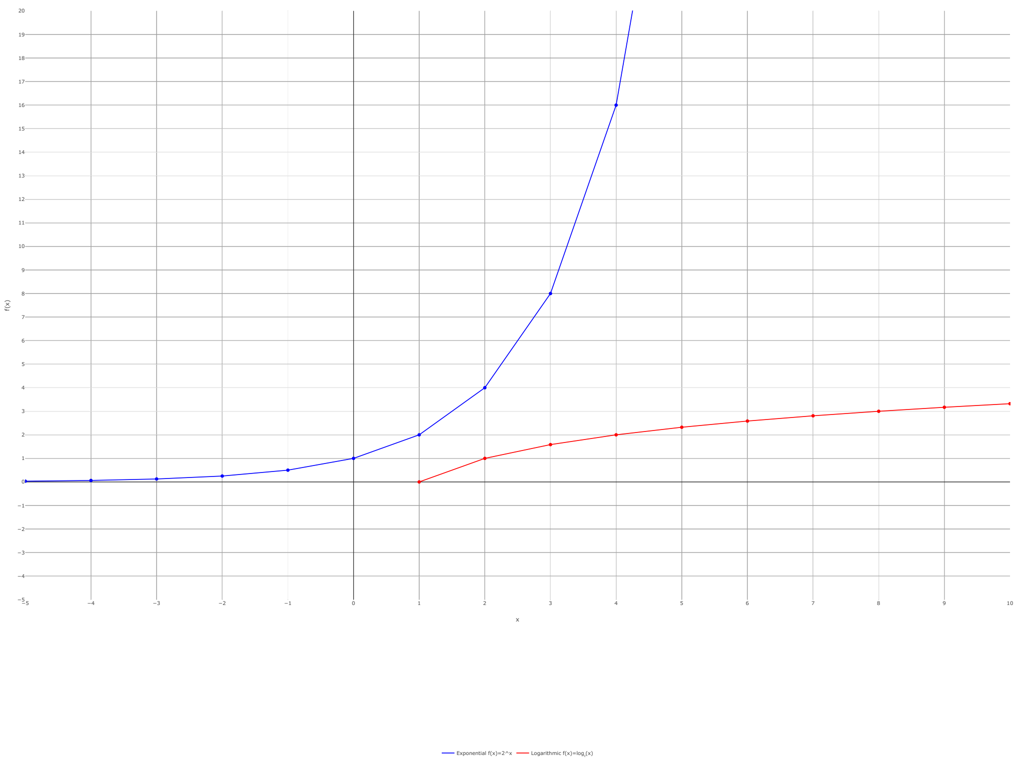

Figure 3.4: Exponential and Logarithmic Functions

3.2.3 Trigonometric Functions

The **trigonometric functions** $f(x) = \sin x, \cos x, \tan x$ describe the relationships between angles and the unit circle. They are fundamental in mathematics, physics, and engineering because they naturally model **oscillations, waves, and circular motion**. These functions are widely used in areas such as signal processing, alternating current circuits, sound and light waves, and applied fields like surveying and mining for modeling cyclic or repetitive patterns (see Figure Figure 3.5).

- **Sine ($\sin x$):** Range $[-1, 1]$, period 2π, zeros at $0°, 180°, 360°$.
  Special values: $\sin 30° = \frac{1}{2},\ \sin 45° = \frac{\sqrt{2}}{2},\ \sin 60° = \frac{\sqrt{3}}{2}$
- **Cosine ($\cos x$):** Range $[-1, 1]$, period 2π, zeros at $90°, 270°$.
  Special values: $\cos 30° = \frac{\sqrt{3}}{2},\ \cos 45° = \frac{\sqrt{2}}{2},\ \cos 60° = \frac{1}{2}$
- **Tangent ($\tan x$):** Period π, undefined at $90°, 270°$.
  Special values: $\tan 30° = \frac{1}{\sqrt{3}},\ \tan 45° = 1,\ \tan 60° = \sqrt{3}$

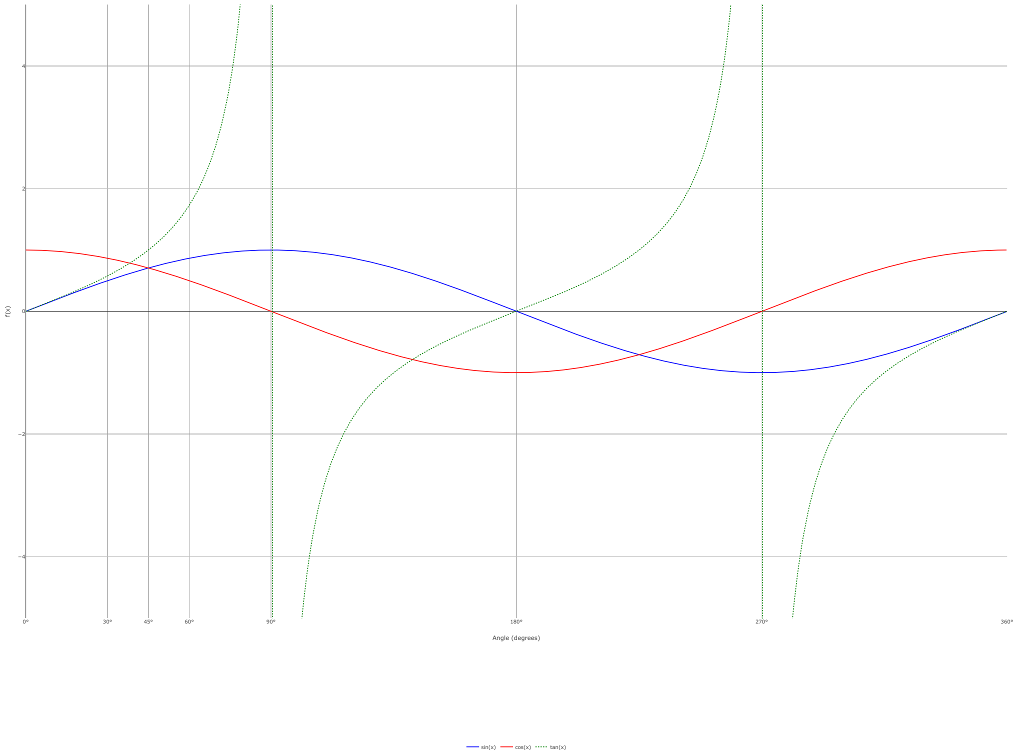

Figure 3.5: Trigonometric Functions: sin(x), cos(x), tan(x) with Special Angles

In trigonometry, certain angles are called special angles because their sine, cosine, and tangent values can be expressed in simple radical forms. These angles — such as $0°, 30°, 45°, 60°$, and $90°$ — are frequently used in mathematics, physics, and engineering for simplifying calculations.

3.3 Properties

Functions can be understood not only from their formulas, but also from the **properties** they possess. These properties describe how inputs and outputs are related, and how the function behaves across its domain and codomain (Figure Figure 3.6). Understanding these characteristics helps in identifying whether a function is one-to-one, onto, continuous, or monotone, which are fundamental concepts in both pure and applied mathematics.

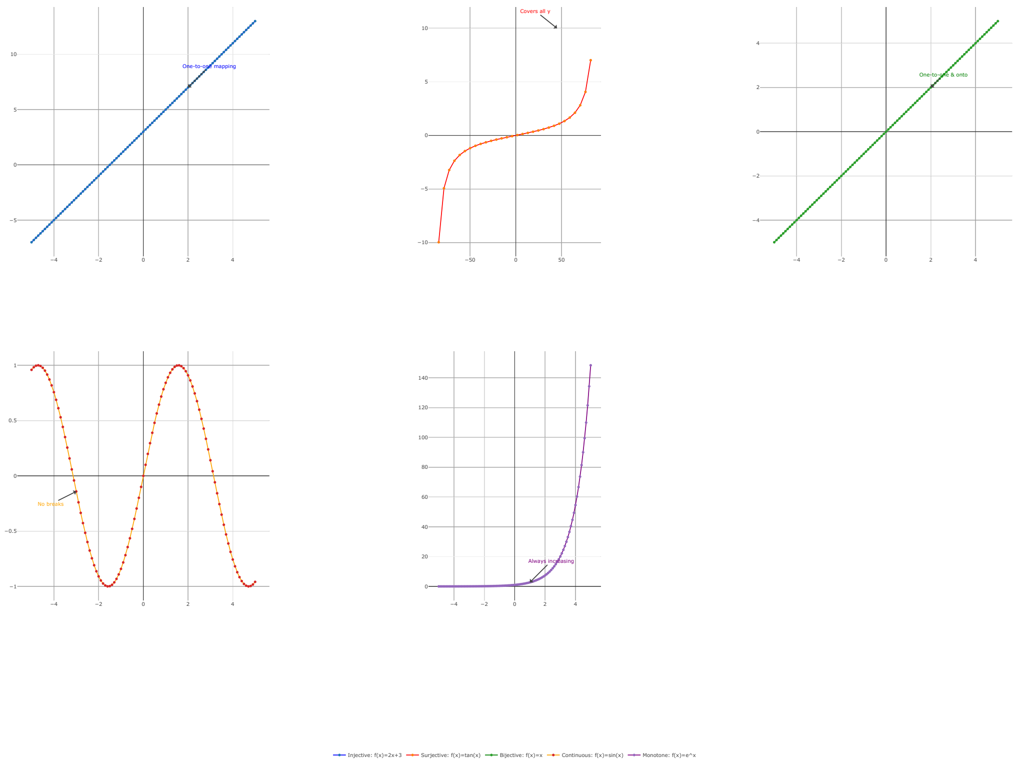

Figure 3.6: Function Properties: Injective, Surjective, Bijective, Continuous, Monotone

3.3.1 Injective (One-to-One)

A function is called **injective** if different inputs always produce different outputs. In other words, no two distinct values in the domain are mapped to the same value in the codomain. For example, $f(x) = 2x + 3$ is injective because every input corresponds to a unique output, whereas $f(x) = x^2$ is not injective over the real numbers, since both 2 and -2 map to the same value, 4. Understanding injective functions is important in applications where each output must correspond to a unique condition, such as tracking ore quality measurements in mining.

3.3.2 Surjective (Onto)

A function is **surjective** if every element in the codomain is "covered" by the function, meaning each possible output has at least one pre-image in the domain. For instance, $f(x) = x^3$ from $\mathbb{R}$ to $\mathbb{R}$ is surjective, while $f(x) = e^x$ is not surjective over all real numbers because it cannot produce negative values. Surjective functions are useful when it is essential that all potential outcomes are achievable, such as ensuring full coverage of production or resource allocation scenarios.

3.3.3 Bijective

When a function is both injective and surjective, it is **bijective**, establishing a perfect one-to-one correspondence between domain and codomain. This means **each element in the domain is paired with exactly one unique element in the codomain, and every element in the codomain is covered**. Every output comes from exactly one input, and an inverse function always exists, allowing us to reverse the mapping easily. For example, $f(x) = x + 5$ is bijective, because no two inputs give the same output and every possible output can be reached.

In practice, bijective functions are valuable in simulations and data transformations, where each output needs to be traced back to a unique input without ambiguity. **They ensure that information is neither lost nor duplicated, making processes reversible and predictable.**

3.3.4 Continuous

A function is **continuous** if its graph can be drawn without lifting the pen. Formally, f is continuous at $x = c$ if $\lim_{x \to c} f(x) = f(c)$. An example is $f(x) = \sin x$, continuous for all real numbers, while $f(x) = 1/x$ is discontinuous at $x = 0$. Continuity is crucial for modeling systems with predictable behavior, such as smooth motion of machinery or fluid flow in mining operations.

3.3.5 Monotone

Functions can also be **monotone**, consistently increasing or decreasing. A monotone increasing function ensures that larger inputs always produce larger outputs, while a monotone decreasing function produces smaller outputs for larger inputs. For example, $f(x) = 2x$ is monotone increasing, while $f(x) = -x$ is monotone decreasing. Monotone functions simplify analysis and optimization, for instance in predicting total ore extracted over time or planning production rates efficiently.

Table 3.2: Key Concepts of Functions

Key Concept	Description	Example / Application
Definition	Maps each x in domain to a unique y in range	$f : x \mapsto y$
Domain and Range	Domain = all possible inputs, Range = all outputs	$x \in [0, 10]$, $f(x) \in [0, 100]$
Linear Function	Straight-line relationship	$f(x) = mx + b$
Quadratic Function	Parabolic relationship	$f(x) = ax^2 + bx + c$
Polynomial Function	Sum of powers of x	$f(x) = a_n x^n + \cdots + a_0$

3.4 Summary

Functions describe relationships between variables, showing how one quantity changes with another. They are fundamental in calculus as they provide **mathematical models** for dynamic systems, patterns, and processes across science, engineering, and economics. A proper understanding of functions requires knowledge of their **domain, range, types, and behavior**. An overview of the key concepts, descriptions, and applications is given in Table 3.2.

3.5 Youtube

```
\newline \href{https://youtu.be/cj8fZb5CS5A}{Click here to watch the video}
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3.6 Applications

References

Table 3.3: Applications of Functions in Real Life and Mining

	Description	Example_Function	Example_Output
Science & Engineering			
Science & Engineering	Modeling position of moving object	$s(t) = 5t$	At t=3 s → s(3)=15 m
Science & Engineering	Modeling velocity of moving object	$v(t) = 2t$	At t=4 s → v(4)=8 m/s
Economics & Finance			
Economics & Finance	Supply-demand curves	$P(x) = 50 + 2x$	Selling 10 items → P(10)=70
Economics & Finance	Compound interest calculation	$A(t) = P(1 + r)^t$	Principal $1000, 5% annual, 3 years → $1157.63
Daily Life			
Daily Life	Temperature conversion	$F(C) = 9/5C + 32$	25°C → 77°F
Daily Life	Daily spending tracking	$S(d) = 10d + 5$	Day 7 → $75
Mining & Resources			
Mining & Resources	Ore production rate	$Q(t) = 1000 + 50t$	After 5 days → Q(5)=1250 tons
Mining & Resources	Mineral concentration	$C(x) = 0.8x + 5$	x=10 → C(10)=13%
Mining & Resources	Operational cost	$Cost(q) = 5000 + 20q$	Produce 100 units → Cost(100)=$7000
Mining & Resources	Heap volume	$V(A) = 2A + 100$	Area=50 m² → V(50)=200 m³

Chapter 4

Operations on Functions

In the previous chapter Essentials of Functions, we explored the foundational concepts of functions—how they relate inputs to outputs, their domains and ranges, and the different types such as linear, quadratic, exponential, and logarithmic functions. Understanding these basic properties allows us to describe mathematical and real-world relationships systematically.

In this chapter, we extend that understanding by studying **Operations on Functions**, which involve combining two or more functions to create a new one. These operations allow us to model more complex relationships, solve optimization problems, and represent multi-step processes commonly found in engineering, business, and natural sciences. Essentially, operations on functions form the bridge between *simple mathematical expressions* and *real-world systems modeling*. Through addition, subtraction, multiplication, division, and composition of functions, we can represent how different processes interact with one another — for example, how production cost depends on both labor efficiency and material usage, or how temperature change affects reaction rates in thermodynamics.

This Figure 4.1 will cover:

- The **definition** of operations on functions
- The **types** of operations, including addition, subtraction, multiplication, division, and composition
- The **purpose and importance** of using these operations in modeling and analysis
- **Applications** in various fields such as engineering, mining, business, and metallurgy

By the end of this chapter, you will be able to combine functions effectively and interpret their interactions within real-world contexts.

4.1 Definition

Operations on functions involve combining two or more functions to create a new one, much like mixing ingredients to make a new recipe. Imagine you have two machines — one that squeezes

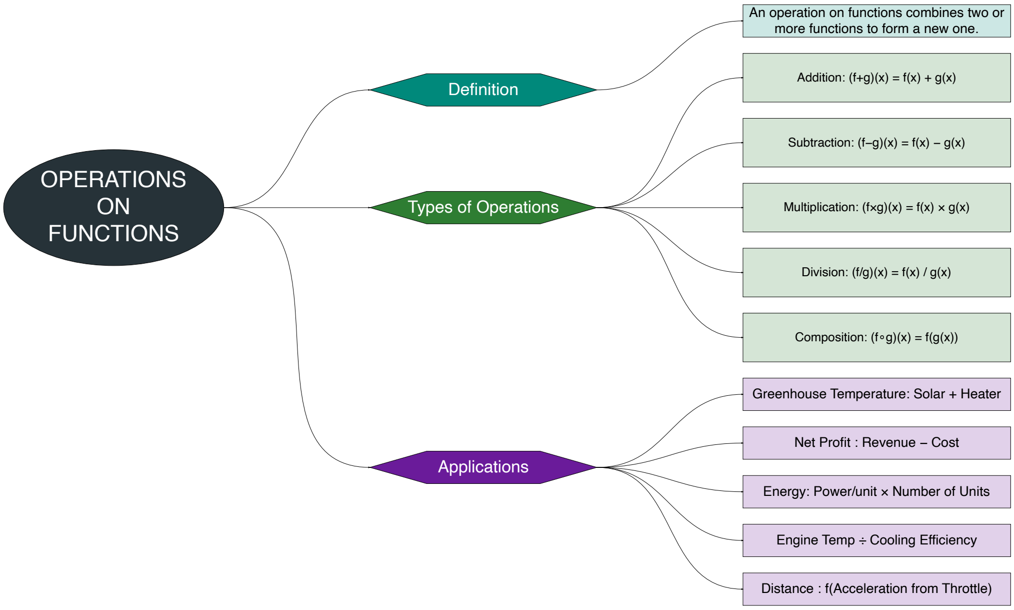

Figure 4.1: Operations on Functions with Real-World Examples

oranges into juice and another that adds sugar and ice. When you connect them, you get a new machine that produces sweet orange juice. Likewise, in mathematics, when two or more functions are combined (through addition, subtraction, multiplication, division, or composition), they create a new function with new behavior and properties.

4.2 Types of Operations

Mathematical operations on functions allow us to combine or manipulate functions to form new ones. Functions can be combined or manipulated in several ways to form new functions. These operations are essential in modeling, analysis, and problem-solving in mathematics, science, engineering, and business.

4.2.1 Addition

Addition is a fundamental operation on functions, where the outputs of two functions are combined for the same input to produce a new function. **Think of it like this:** imagine two machines one makes orange juice $f(x)$ and the other makes sugar syrup $g(x)$. By combining them, you get a **new machine producing sweet orange juice**, which reflects the sum of their individual outputs.

> **Example:**
>
> Suppose we have two functions:

$$\begin{aligned} f(x) &= 2x + 3, \\ g(x) &= x^2. \end{aligned} \tag{4.1}$$

Solution

To find the sum of the two functions (4.1), we add their corresponding expressions as follows:

$$\begin{aligned} (f+g)(x) &= f(x) + g(x) \\ &= (2x+3) + x^2 \\ &= x^2 + 2x + 3 \end{aligned} \tag{4.2}$$

Equation (4.2) shows that adding two functions produces a new function that combines the effects of both.

Visualization

To gain a clearer understanding of the process of function addition, the following visualization illustrates how the functions $f(x) = 2x + 3$ and $g(x) = x^2$ interact. By plotting both functions along with their sum $(f+g)(x)$, we can observe how the resulting function combines the linear growth of $f(x)$ with the quadratic behavior of $g(x)$.

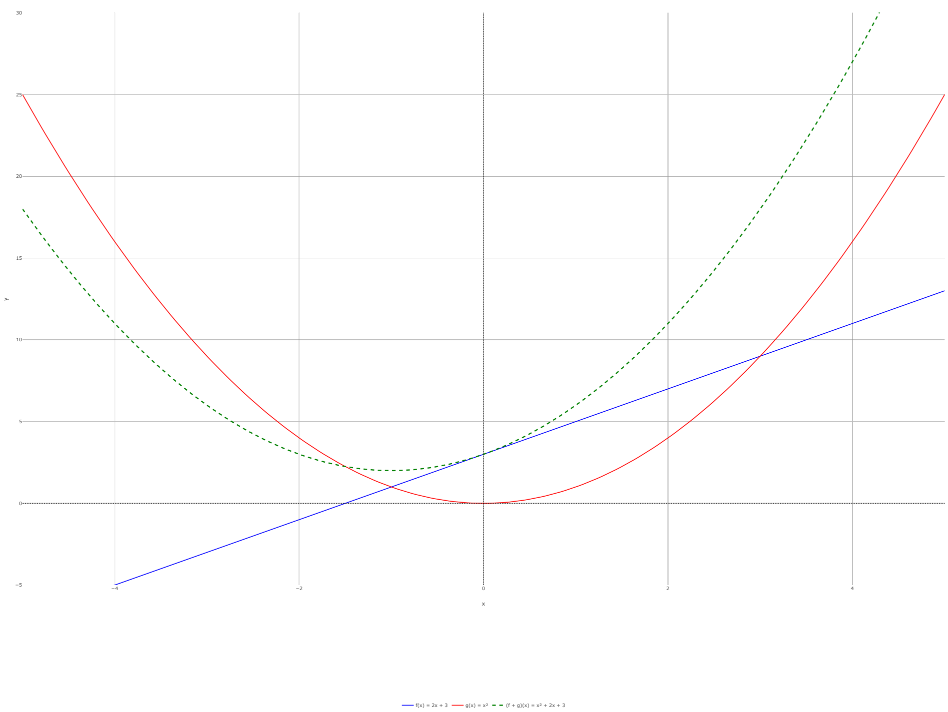

Figure 4.2: Visualisasi Penjumlahan Fungsi: f(x) = 2x + 3 dan g(x) = x^2

4.2.2 Subtraction

Subtraction is similar to addition but involves finding the difference between the outputs of two functions for the same input. **Analogy:** imagine the orange juice machine $f(x)$ and the sugar syrup machine $g(x)$. Subtracting them gives a **machine producing juice with reduced sweetness**, reflecting the difference of their contributions.

Example:

Suppose we have two functions:

$$\begin{aligned} f(x) &= 5x, \\ g(x) &= 2x. \end{aligned} \tag{4.3}$$

Solution

To find the difference of the two functions (4.3):, we subtract their corresponding expressions as follows:

$$\begin{aligned} (f-g)(x) &= f(x) - g(x) \\ &= 5x - 2x \\ &= 3x \end{aligned} \tag{4.4}$$

Equation (4.4) shows that the difference between two functions produces a new function that represents the subtraction of $g(x)$ from $f(x)$.

Visualization

Before visualizing the subtraction of two functions, it is essential to understand that this operation involves determining the difference between their respective output values for each corresponding input. This process reveals how one function behaves relative to another, allowing us to analyze the resulting change or rate of difference between them.

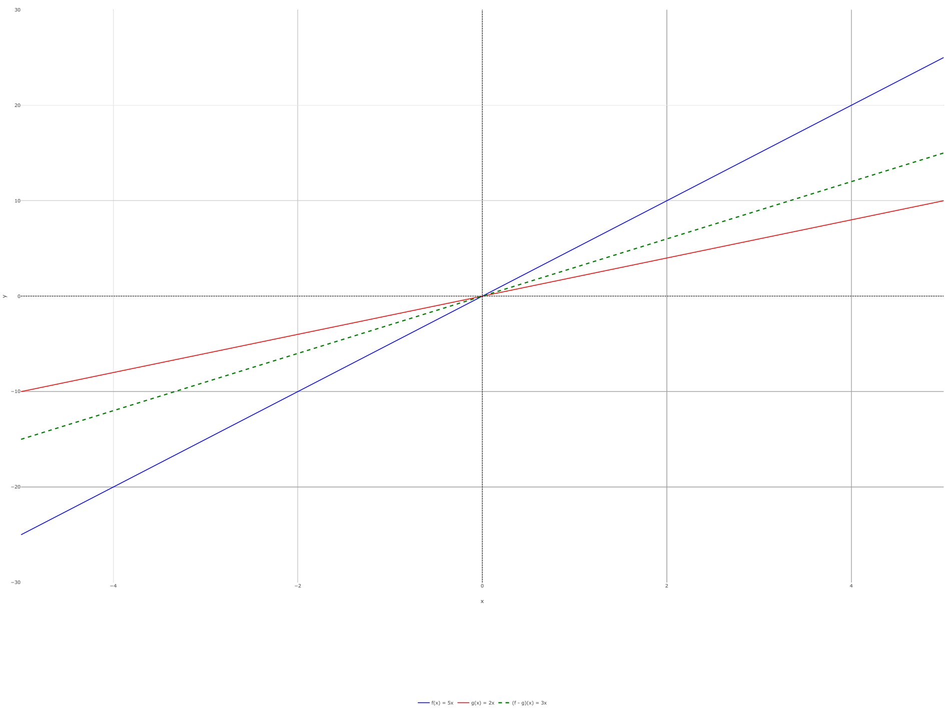

Figure 4.3: Graphical Representation of Function Subtraction: f(x) = 5x and g(x) = 2x

4.2.3 Multiplication

Multiplication combines two functions by multiplying their outputs for the same input. Analogy: imagine the orange juice machine $f(x)$ and a sugar syrup machine $g(x)$. Multiplying them produces a **machine that outputs the product of juice and syrup concentration**, enhancing the effect multiplicatively.

Example:

Suppose we have two functions:

$$\begin{aligned} f(x) &= x + 1, \\ g(x) &= 2x. \end{aligned} \tag{4.5}$$

Solution

To find the product of the two functions (4.5), we multiply their corresponding expressions as follows:

$$\begin{aligned}(f \cdot g)(x) &= f(x) \cdot g(x) \\ &= (x+1)(2x) \\ &= 2x^2 + 2x\end{aligned} \tag{4.6}$$

Equation 4.6 shows that multiplying two functions combines their outputs multiplicatively, producing a new function that scales both effects together.

Visualization

Before visualizing the multiplication of two functions, it is important to note that this operation involves combining the output values of both functions through pointwise multiplication.
The resulting product function illustrates how the interaction between the two functions amplifies or reduces their respective magnitudes across the domain.

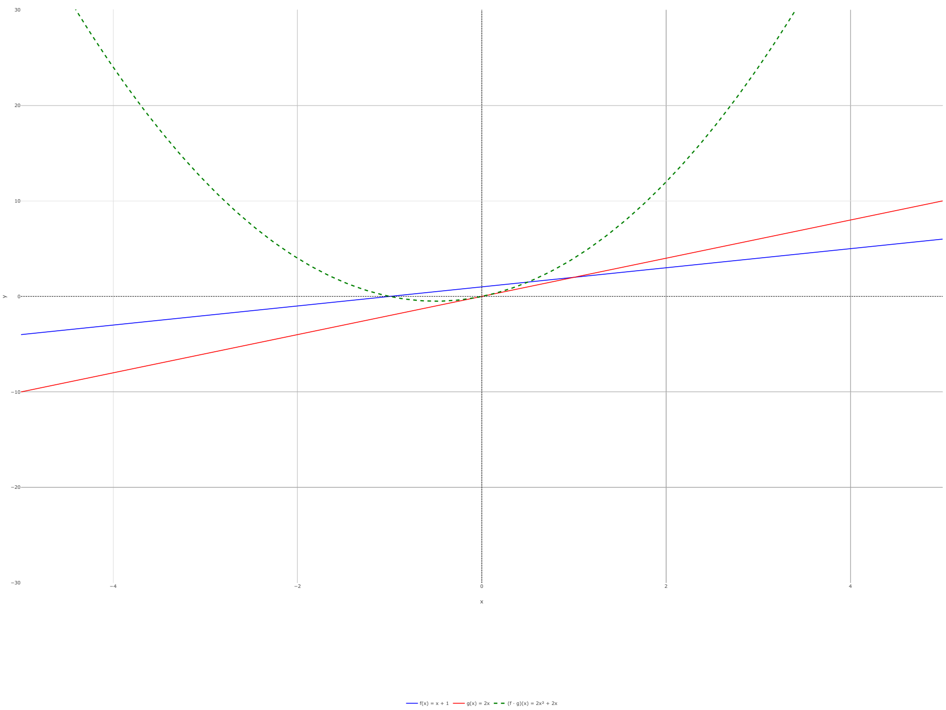

Figure 4.4: Graphical Representation of Function Multiplication: f(x) = x + 1 and g(x) = 2x

4.2.4 Division

Division forms a new function by dividing the output of one function by another, provided the denominator is not zero. Analogy: imagine the orange juice machine $f(x)$ and a syrup machine $g(x)$. Dividing them produces a **machine that controls sweetness ratio**, balancing the two contributions.

Example:

Suppose we have two functions:

$$\begin{aligned} f(x) &= x^2 + 4, \\ g(x) &= 2x. \end{aligned} \tag{4.7}$$

Solution

To find the quotient of the two functions 4.7, we divide their corresponding expressions as follows:

$$\begin{aligned} \left(\frac{f}{g}\right)(x) &= \frac{f(x)}{g(x)}, \quad g(x) \neq 0 \\ &= \frac{x^2+4}{2x} \\ &= \frac{x^2}{2x} + \frac{4}{2x} \\ &= \frac{x}{2} + \frac{2}{x}, \quad x \neq 0 \end{aligned} \tag{4.8}$$

Equation 4.8 shows that dividing two functions produces a new function that represents the ratio of $f(x)$ to $g(x)$, provided $g(x) \neq 0$.

Visualization

Division of functions involves determining the ratio of two corresponding outputs, provided that the denominator function is non-zero. This operation produces a new function that illustrates how the numerator's growth rate compares to that of the denominator across the domain.

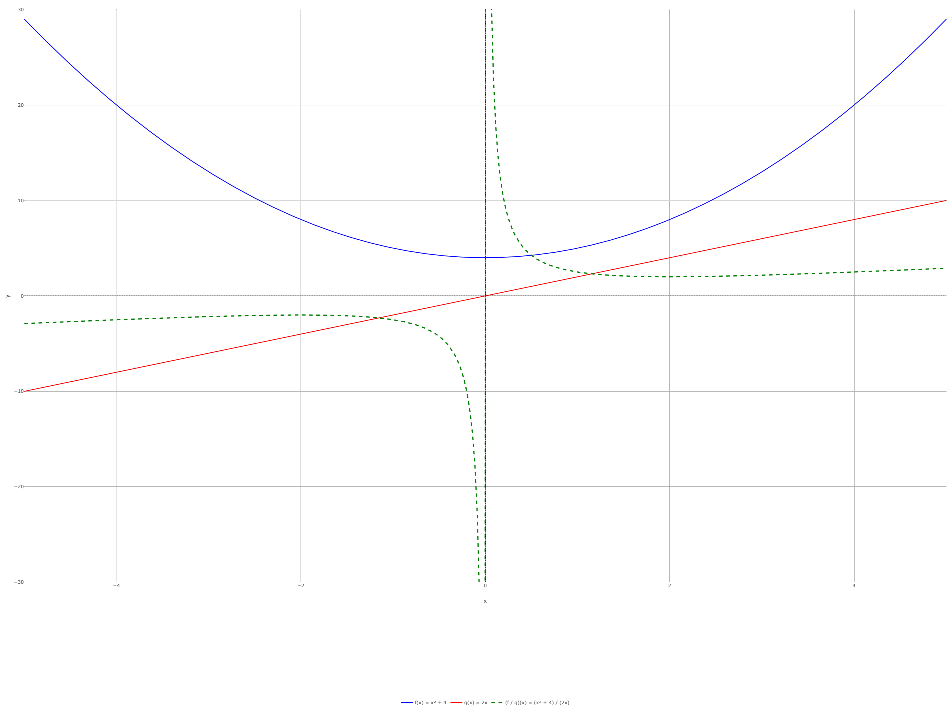

Figure 4.5: Graphical Representation of Function Division: f(x) = x² + 4 and g(x) = 2x

4.2.5 Composition

Composition involves using the output of one function as the input of another. Analogy: imagine the orange juice machine $g(x)$ feeds its juice into a blender $f(x)$. The **new machine** produces a blended juice whose output depends on both machines in sequence.

Example:

Suppose we have two functions:

$$\begin{aligned} f(x) &= x + 1, \\ g(x) &= x^2. \end{aligned} \tag{4.9}$$

Solution

To find the composition of the two functions 4.9, we substitute $g(x)$ into $f(x)$ as follows:

$$\begin{aligned} (f \circ g)(x) &= f(g(x)) \\ &= g(x) + 1 \\ &= x^2 + 1 \end{aligned} \tag{4.10}$$

Equation 4.10 shows that the composition $(f \circ g)(x)$ produces a new function where the output of $g(x)$ becomes the input of $f(x)$.

Visualization

Function composition involves applying one function to the result of another. In this case, the output of $g(x)$ becomes the input to $f(x)$, forming a new composite function $(f \circ g)(x) = f(g(x))$. This operation demonstrates how sequential transformations affect the shape of the resulting function.

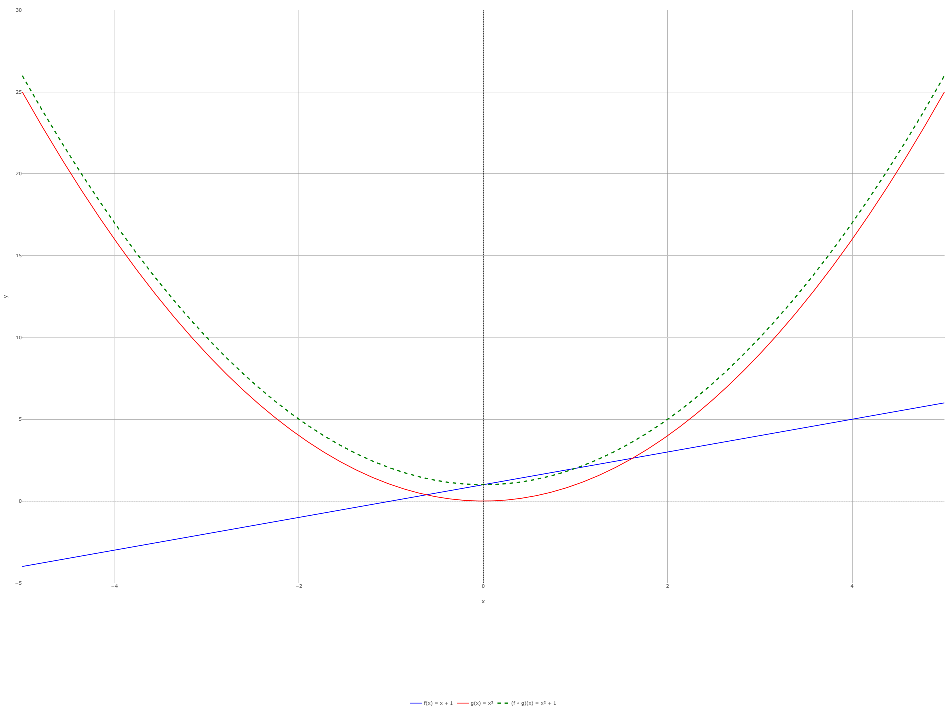

Figure 4.6: Visualization of Function Composition: f(x) = x + 1 and g(x) = x²

4.3 Real-World Applications

4.3.1 A Car Engine

A car engine generates heat while operating. Without a cooling system, the engine temperature rises continuously, which can damage components. We want to **model the actual engine temperature** considering the effect of the cooling system. This is a real example of **combining functions**, where multiple factors are represented in a single equation.

Function Definitions

1. **Engine Temperature without Cooling**

The engine generates heat over time: $f(t) = 2t^2 + 30$

- t: time the engine runs (minutes)
- $f(t)$: engine temperature (°C) without cooling
- Interpretation: temperature rises **quadratically** as heat increases faster over time.

2. **Cooling System Efficiency**

The cooling system reduces heat, but efficiency decreases over time: $g(t) = 0.5t + 1$

- $g(t)$: cooling factor (larger = more effective)
- As t increases, cooling efficiency becomes relatively weaker compared to engine heat.

Combining Functions

The actual engine temperature $T(t)$ can be modeled as **engine temperature divided by cooling efficiency**:

$T(t) = \frac{f(t)}{g(t)}$

Substitute the functions:

$T(t) = \frac{2t^2+30}{0.5t+1}$

Simplification

Step by step:

1. Factor out constants where possible: $T(t) = \frac{2(t^2+15)}{0.5t+1}$
2. Multiply numerator and denominator by 2 to simplify: $T(t) = \frac{4(t^2+15)}{t+2}$

Domain restriction: $g(t) \neq 0 \Rightarrow t \neq -2$ (physically irrelevant since $t \geq 0$).

Interpretation

1. At small t, cooling is effective → engine temperature remains moderate.
2. As t increases, the effect of cooling diminishes relative to engine heat → temperature rises faster.
3. This model shows the **interaction between two factors** (engine heat and cooling efficiency) mathematically, not just from empirical observation.

Applications

- **Cooling System Design:** Estimate the cooling capacity needed to maintain safe temperatures.
- **Engine Failure Prediction:** Determine when engine temperature may become too high.
- **Simulation & Optimization:** Test various operation times, efficiency, or enhanced cooling without real-world experiments.

4.3.2 Energy and Metallurgy

In metallurgy, **energy efficiency and chemical reactions** are critical for designing furnaces and reactors. By **combining functions**, engineers can model **heat generation, fuel consumption, and reaction progress**, and optimize for efficiency or yield.

Combustion in a Blast Furnace

1. **Heat Generated by Fuel Combustion**

The heat produced Q_f depends on the amount of fuel burned m_f:
$Q_f(m_f) = 30m_f$ (MJ/kg of fuel)

- m_f: mass of fuel burned (kg)
- Q_f: total heat generated (MJ)

2. **Heat Required for Ore Reduction**

The heat needed to reduce iron ore Q_r depends on the mass of ore m_o:
$Q_r(m_o) = 25m_o$ (MJ/kg of ore)

- m_o: mass of ore processed (kg)
- Q_r: energy required for the reduction reaction (MJ)

Energy Balance and Efficiency

The **process efficiency** η can be modeled as the ratio of useful energy to energy supplied:
$\eta(m_f, m_o) = \frac{Q_r(m_o)}{Q_f(m_f)} = \frac{25m_o}{30m_f}$

- To achieve **100% energy efficiency**, set $\eta = 1$: $25m_o = 30m_f \implies m_f = \frac{25}{30}m_o = 0.833m_o$
- This means for every **1 kg of ore**, we need **0.833 kg of fuel** for an ideal energy balance.

Table 4.1: Real-World Applications of Function Operations

Operation	Mathematical_Form	Real_World_Example	Description
Addition	$(f+g)(x) = f(x) + g(x)$	Greenhouse temperature: solar heating + heater	Total temperature = contributions from multiple heat sources
Subtraction	$(f-g)(x) = f(x) - g(x)$	Net profit: revenue − cost	Profit = Revenue minus Cost, representin gain
Multiplication	$(f \cdot g)(x) = f(x) \cdot g(x)$	Energy production: power per unit × number of units	Total energy produced = power output × number of units
Division	$(f/g)(x) = f(x)/g(x)$	Engine temperature per cooling efficiency: heat ÷ cooling factor	Actual temperature considering cooling efficiency
Composition	$(f \circ g)(x) = f(g(x))$	Car speed control: distance = f(acceleration from throttle)	Output of one process depends on the out another function

Interpretation

- $\eta < 1$: fuel is **overused** → excess heat, higher cost.
- $\eta > 1$: fuel is **insufficient** → reaction incomplete, lower yield.
- By combining Q_f and Q_r functions, we can **optimize fuel usage** and improve furnace efficiency.

Applications

- **Metallurgy:** Optimize fuel consumption for smelting or reduction reactions.
- **Chemical Engineering:** Balance energy inputs for industrial reactors.
- **Sustainable Design:** Reduce fuel costs and emissions by maximizing efficiency.

4.3.3 Others

Functions are widely used to model real-world systems. Often, we need to **combine multiple functions** to understand how different factors interact. The Table 4.1 provides **practical examples for each type of function operation**—addition, subtraction, multiplication, division, and composition—along with their interpretations.

References

Chapter 5

Limits of Functions

In the previous chapter Operations on Functions, we explored how to combine functions through addition, subtraction, multiplication, division, and composition. These operations allowed us to model multi-step processes and represent interactions between different mathematical and real-world quantities.

In this chapter (See, Figure 5.1), we extend that understanding by studying **Limits of Functions**, which analyze the behavior of functions as their inputs approach specific points or infinity. Limits are a fundamental concept in calculus, forming the foundation for continuity, derivatives, and integrals. They allow us to rigorously describe how functions behave near critical points, handle discontinuities, and model phenomena where changes are instantaneous or unbounded.

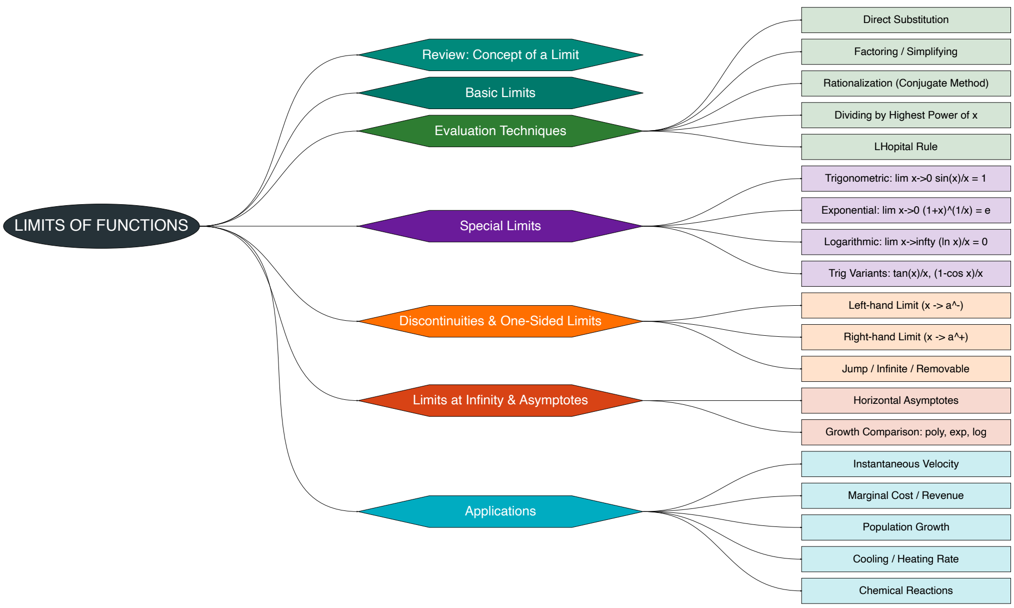

Figure 5.1: Limits of Functions — Concept, Techniques, and Applications

5.1 Review

Before diving into formal definitions, let's use an analogy to understand what a **limit** is. Imagine you are driving a car toward a stop sign. As you get closer, your speed gradually decreases, approaching zero. You never jump from a high speed to zero instantly—the approach is smooth and continuous. Similarly, in mathematics, a **limit** describes the value a function is approaching as the input gets closer to a particular point, even if the function doesn't exactly reach that value at the point itself.

Watch here: The limit of a function

This animation (Figure 5.2) illustrates this concept by showing a car approaching a stop sign over time. The distance to the stop sign is represented by the function $d(t) = 10e^{-0.3t}$, which decreases exponentially, approaching zero as time increases. The visualization highlights how the car's distance diminishes gradually, making the idea of a limit intuitive and easy to grasp.

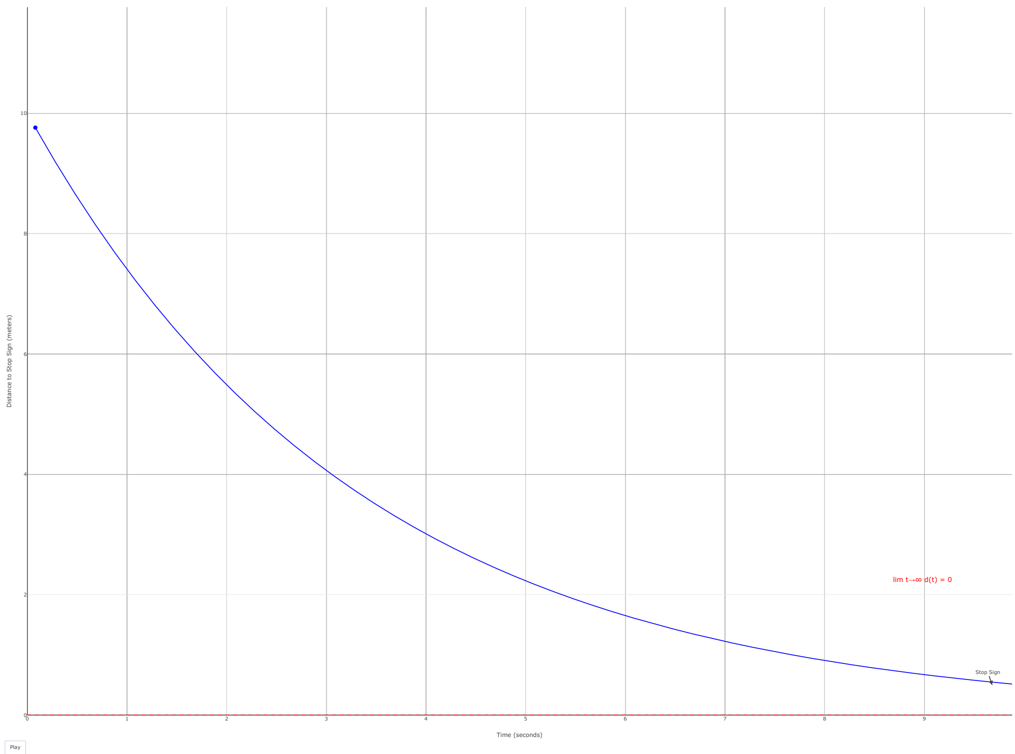

Figure 5.2: Animated Simulation: Car Approaching Stop Sign

⚠ Explanation of the Graph: $(d(t) = 10e^{-0.3t}$

The graph (Figure 5.2) illustrates the concept of a **mathematical limit**, showing how the car's distance to a stop sign decreases over time and approaches zero without ever becoming negative.

- **Blue line:** shows the car's distance to the stop sign over time.
- **Blue point:** represents the car's position at each moment, moving down along the function.
- **Red dashed line:** the stop sign, acting as a **horizontal asymptote** at $y = 0$.
- **Function behavior:**
  - Exponentially decreasing: the distance drops quickly at first, then gradually approaches zero.
  - Always decreasing → the car never moves away from the stop sign.
- **Limit:** $\lim_{t\to\infty} d(t) = 0$ → the car approaches the stop sign but never passes it.

5.2 Basic Limits

The limit of a function describes the behavior of the function as the input approaches a particular point. Formally, the limit of a function $f(x)$ as $x \to a$ is written as:

$$\lim_{x\to a} f(x) = L$$

This means that as x gets closer to a, $f(x)$ gets closer to L.

Example:

$$f(x) = 2x + 3$$

$$\lim_{x\to 1} f(x) = 2(1) + 3 = 5$$

Notes:

- The limit **does not have to equal the function's value** at that point.
- If $f(a)$ exists, the limit may equal $f(a)$. If not, the limit can still exist.

5.3 Evaluation Techniques

In calculus, understanding how to evaluate limits is essential for analyzing the behavior of functions near specific points or as they approach infinity. Various techniques are used to simplify complex expressions, handle indeterminate forms, and determine whether a limit exists or diverges. These methods form the foundation for deeper topics such as continuity, derivatives, and integrals.

The following section presents several key techniques commonly used in evaluating limits, including direct substitution, factoring, rationalization, simplifying complex fractions, and applying special limit theorems. Each technique helps to reveal the underlying structure of a function and ensures accurate computation of limits in different contexts.

5.3.1 Direct Substitution

If $f(x)$ is continuous at $x = a$, then:

$$\lim_{x \to a} f(x) = f(a)$$

Example:

$$\lim_{x \to 2} (3x + 1) = 3(2) + 1 = 7$$

5.3.2 Factoring and Simplifying

Used when substitution gives an indeterminate form $0/0$.

Example:

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$$

Factor the numerator:

$$\frac{(x-2)(x+2)}{x-2}$$

Cancel $(x - 2)$, then substitute:

$$\lim_{x \to 2} (x + 2) = 4$$

5.3.3 Rationalizing

Rationalizing (Conjugate Method) used when square roots cause indeterminate forms.

Example:

$$\lim_{x \to 0} \frac{\sqrt{x+4} - 2}{x}$$

Multiply by the conjugate:

$$\frac{\sqrt{x+4} - 2}{x} \cdot \frac{\sqrt{x+4} + 2}{\sqrt{x+4} + 2} = \frac{x}{x(\sqrt{x+4} + 2)} = \frac{1}{\sqrt{x+4} + 2}$$

Then substitute $x = 0$:

$$\lim_{x \to 0} \frac{1}{\sqrt{4} + 2} = \frac{1}{4}$$

5.3.4 Dividing by the Highest Power of x

Used for **limits at infinity** with rational functions.

Example:

$$\lim_{x\to\infty} \frac{3x^2+2}{x^2+5}$$

Divide numerator and denominator by x^2:

$$\frac{3+\frac{2}{x^2}}{1+\frac{5}{x^2}} \to \frac{3}{1} = 3$$

5.3.5 L'Hôpital's Rule

Used for **indeterminate forms** $\frac{0}{0}$ or $\frac{\infty}{\infty}$. If $\lim_{x\to a} \frac{f(x)}{g(x)}$ gives $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Example:

$$\lim_{x\to 1} \frac{x^2-1}{x-1}$$

At $x = 14:

$$\frac{1^2-1}{1-1} = \frac{0}{0}$$

This expression results in an **indeterminate form**, meaning the limit cannot be directly evaluated by simple substitution.

Therefore, differentiate numerator and denominator:

$$\frac{d}{dx}(x^2-1) = 2x, \quad \frac{d}{dx}(x-1) = 1$$

Then:

$$\lim_{x\to 1} \frac{2x}{1} = 2$$

5.4 Special Limits

Special limits are common patterns that appear frequently in calculus. They often involve trigonometric, exponential, or logarithmic functions that have well-known limit results. Understanding these helps simplify many limit problems quickly.

5.4.1 Trigonometric Limit

$\lim_{x\to 0} \frac{\sin x}{x}$

This is one of the most fundamental limits in calculus.

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

Reasoning:

As x approaches 0, both $\sin x$ and x approach 0 at the same rate. Graphically, the sine curve and the line $y = x$ are nearly identical near the origin.

Variants:

$$\lim_{x\to 0} \frac{\tan x}{x} = 1, \quad \lim_{x\to 0} \frac{1-\cos x}{x} = 0$$

5.4.2 Exponential Limit

$\lim\limits_{x\to 0}(1+x)^{1/x}$

This limit defines the **mathematical constant** e (Euler's number).

$$\lim_{x\to 0}(1+x)^{1/x} = e$$

Alternative form:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

Interpretation: This expression arises naturally in compound interest and growth models, it represents continuous growth where compounding occurs infinitely often.

5.4.3 Exponential Functions

For any real number k:

$$\lim_{x\to 0} e^{kx} = 1$$

since $e^{kx} \to e^0 = 1$.

5.4.4 Logarithmic Functions

$$\lim_{x\to 0^+} \ln(1+x) = 0, \quad \lim_{x\to\infty} \frac{\ln x}{x} = 0$$

5.4.5 Trigonometric Functions

$$\lim_{x\to 0} \frac{\sin(kx)}{x} = k, \quad \lim_{x\to 0} \frac{1-\cos(kx)}{x^2} = \frac{k^2}{2}$$

5.5 Discontinuities

5.5.1 One-Sided Limits

Left-hand limit (approaching from the left):

$$\lim_{x\to a^-} f(x) = L$$

The function approaches L as x approaches a from the left ($x < a$).

Right-hand limit (approaching from the right):

$$\lim_{x\to a^+} f(x) = L$$

The function approaches L as x approaches a from the right ($x > a$).

Limit exists if:

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$$

Example:

$$f(x) = \begin{cases} x+2 & x < 1 \\ 3x & x \geq 1 \end{cases}$$

- $\lim_{x\to 1^-} f(x) = 1 + 2 = 3$
- $\lim_{x\to 1^+} f(x) = 3(1) = 3$
- So, $\lim_{x\to 1} f(x) = 3$

5.5.2 Infinite Limits

Infinite Limits (Vertical Asymptote) occur when the function **grows without bound** as it approaches a certain point. We are asking: *"What happens to $f(x)$ as x gets close to some point a?"*

$$\lim_{x\to a} f(x) = \infty \quad \text{or} \quad -\infty$$

Example:

$$f(x) = \frac{1}{(x-2)^2}$$

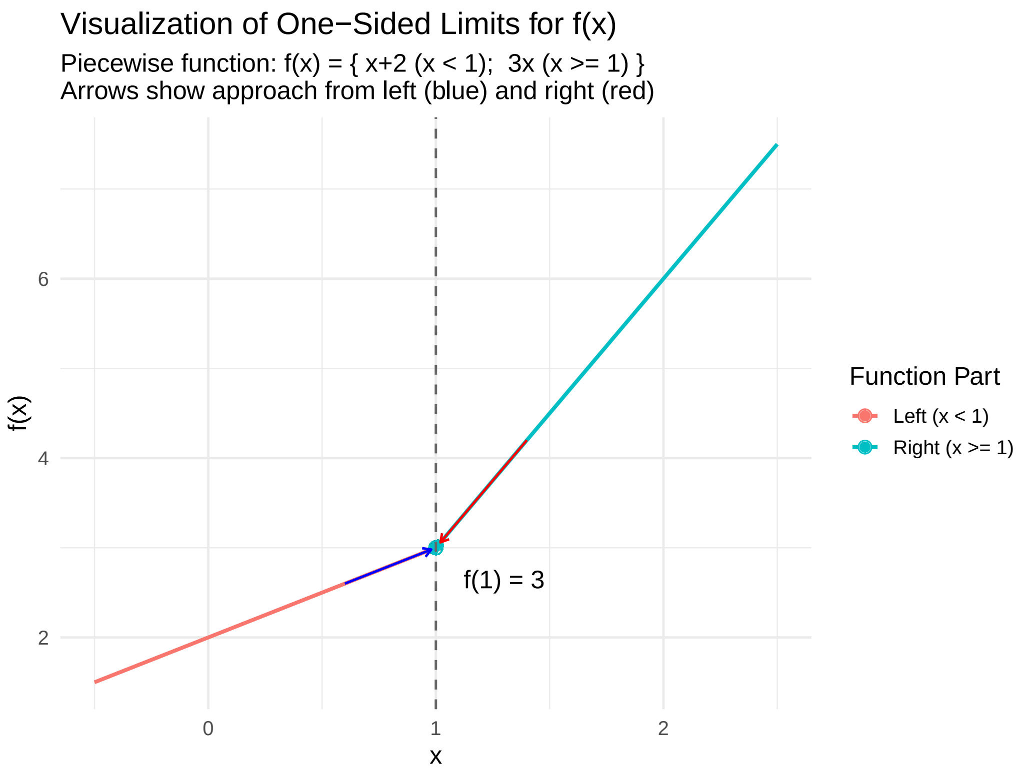

Figure 5.3: Limits of Functions — Concept, Techniques, and Applications

$$\lim_{x \to 2} f(x) = \infty$$

This shows the function **blows up** as x approaches 2.

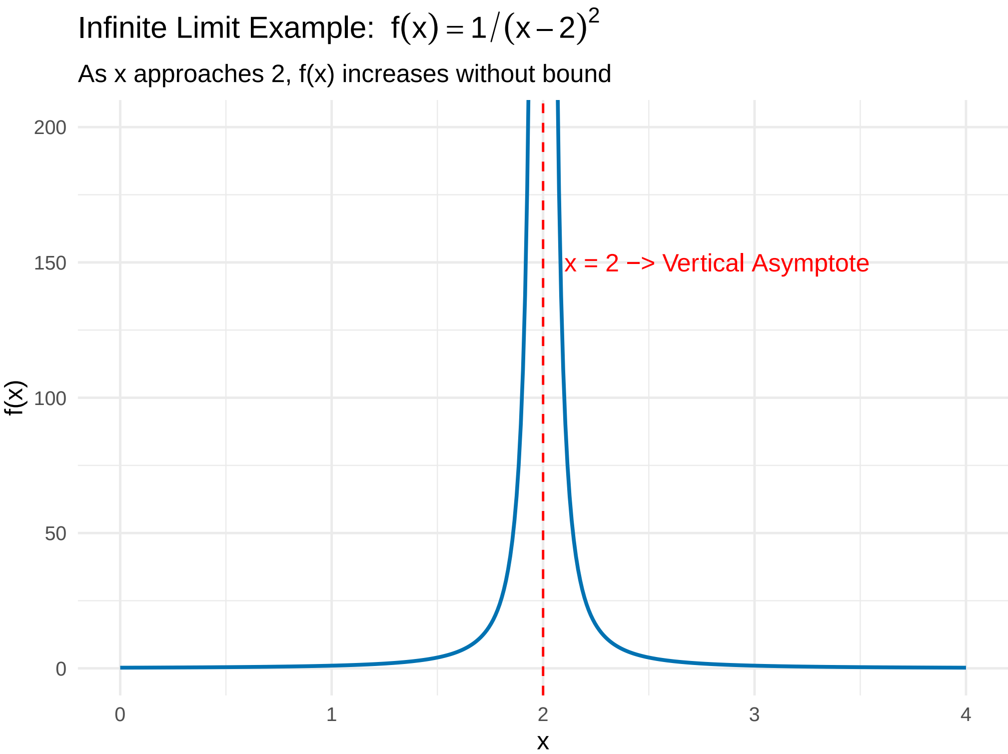

Figure 5.4: Vertical Asymptote

5.5.3 Limits at Infinity

These describe the **end behavior** of a function as x becomes very large ($x \to \infty$) or very small ($x \to -\infty$). We are asking: *"What value does $f(x)$ approach as x goes to infinity?"*

Consider the function:

$$f(x) = \frac{3x^2 + 2}{x^2 + 5}$$

The degree of the numerator $= 2$, and the degree of the denominator $= 2$. $\Rightarrow$ The degrees are **the same**. Take the coefficients of the highest-degree terms: Numerator $= 3$, Denominator $= 1$.

Therefore:

$$\lim_{x \to \infty} f(x) = \frac{3}{1} = 3$$

Step-by-step calculation (dividing by x^2):

$$\frac{3x^2+2}{x^2+5} = \frac{3+\frac{2}{x^2}}{1+\frac{5}{x^2}}$$

As $x \to \infty$, every term containing $\frac{1}{x^2}$ approaches 0. Thus, we are left with:

$$\frac{3}{1} = 3$$

For $x \to -\infty$: The same logic applies, since $\frac{2}{x^2} \to 0$ and $\frac{5}{x^2} \to 0$.
Therefore:

$$\lim_{x \to -\infty} f(x) = 3$$

*Conclusion: **The graph of the function has a** horizontal asymptote** at $y = 3$ in both directions.

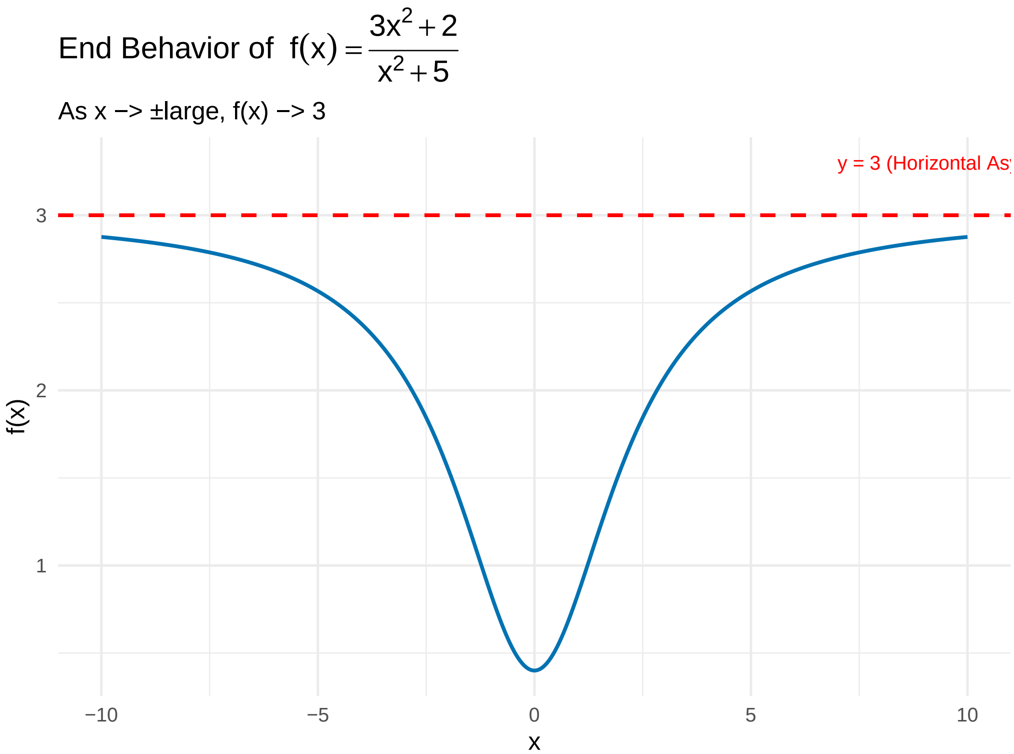

Figure 5.5: Horizontal Asymptote

5.5.4 Growth Comparison

As $x \to \infty$, some functions **grow** much faster than others. Understanding **growth comparison** is essential to:

- Evaluate **limits at infinity**,
- Identify **horizontal asymptotes**, and
- Determine which function **dominates** in complex expressions.

In general, the order of growth from **slowest** to **fastest** is:

$$\ln(x) \ll x^a \ll a^x \ll x! \ll x^x$$

> or verbally: **Logarithmic < Polynomial < Exponential < Factorial < Power Tower**

Function Type	Example	Growth as $x \to \infty$	Notes
Logarithmic	$\ln(x)$	Slowest growth	Increases unboundedly but very slowly
Polynomial	x^2, x^3, x^n	Moderate growth	Dominates logarithmic functions
Exponential	$2^x, e^x, 10^x$	Fast growth	Always beats any polynomial
Factorial	$x!$	Very fast	Super-exponential growth
Power Tower	x^x, a^{x^2}	Extremely fast	Grows faster than all others

5.6 Applications

The concept of instantaneous velocity comes directly from the derivative of a function — a core application of limits and differential calculus. Suppose the position of an object moving along a straight line is given by a function:

$$s = f(t)$$

where:

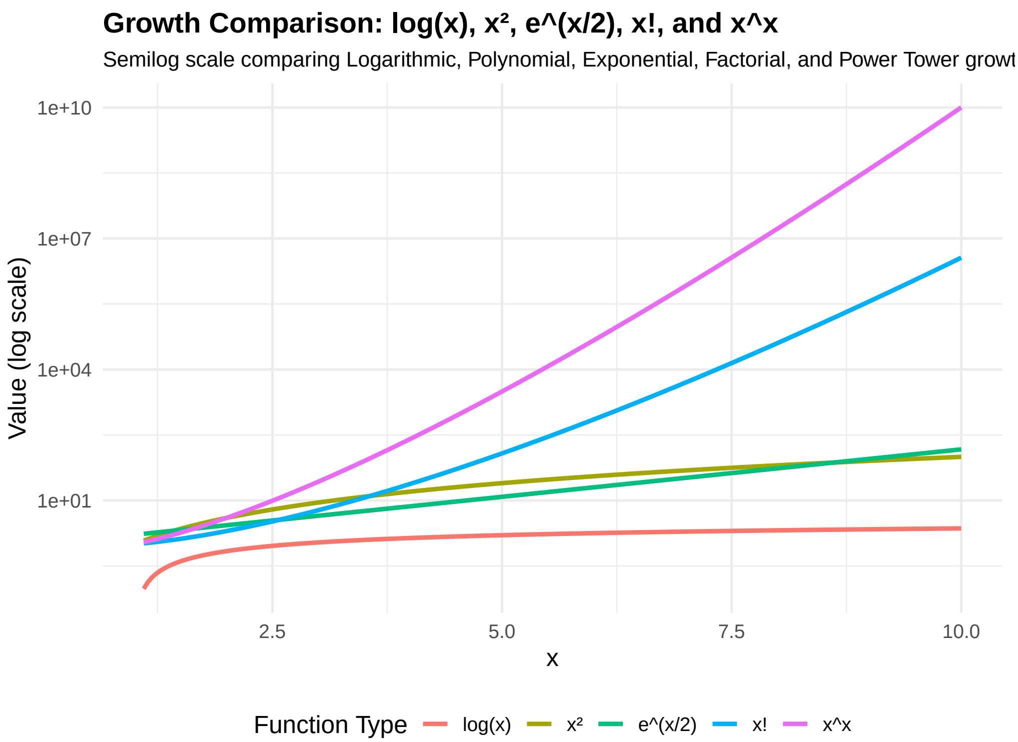

Figure 5.6: Extended Growth Comparison

- s = position (in meters, for example)
- t = time (in seconds)

The **average velocity** of the object between two points in time t and $t + h$ is defined as the **change in position** divided by the **change in time**:

$$v_{\text{avg}} = \frac{\text{change in position}}{\text{change in time}} = \frac{f(t+h) - f(t)}{h}$$

This formula represents the **average rate of change** of the function $f(t)$ over the interval $[t, t+h]$.

- If h is large, v_{avg} gives only a coarse estimate of the object's motion.
- If h is small, v_{avg} becomes a more accurate approximation of how fast the object is moving **at time** t.

To find the **exact velocity at a single instant**, we let the time interval h approach zero. This leads to the **limit definition** of the derivative:

$$v(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$$

Thus, **instantaneous velocity** is defined as the **derivative** of the position function:

$$v(t) = f'(t)$$

Geometrically, the expression

$$\frac{f(t+h) - f(t)}{h}$$

represents the **slope of the secant line** connecting the points $(t, f(t))$ and $(t + h, f(t + h))$ on the graph of $s = f(t)$.

As $h \to 0$, the two points move closer together, and the secant line approaches the **tangent line** at $(t, f(t))$. The slope of this tangent line gives the **instantaneous velocity** at that point.

From a physical standpoint:

- If $v(t) > 0$, the position s increases — the object moves **forward**.
- If $v(t) < 0$, the position s decreases — the object moves **backward**.
- If $v(t) = 0$, the object is **momentarily at rest**.

Thus, the derivative $f'(t)$ provides not only the **speed** but also the **direction** of motion.

Let the position of an object be defined by:

$$f(t) = t^2 + 3t$$

Then the instantaneous velocity is the derivative:

$$v(t) = f'(t) = 2t + 3$$

To find the velocity at $t = 4$ seconds:

$$v(4) = 2(4) + 3 = 11$$

Hence, at $t = 4$ seconds, the instantaneous velocity is **11 m/s**.

If we plot $s = f(t)$, the **instantaneous velocity** at time t is represented by the **slope of the tangent line** to the curve.

As h decreases, the secant line connecting $(t, f(t))$ and $(t + h, f(t + h))$ becomes nearly identical to the tangent line — showing how the **limit process** captures the instantaneous rate of change.

References

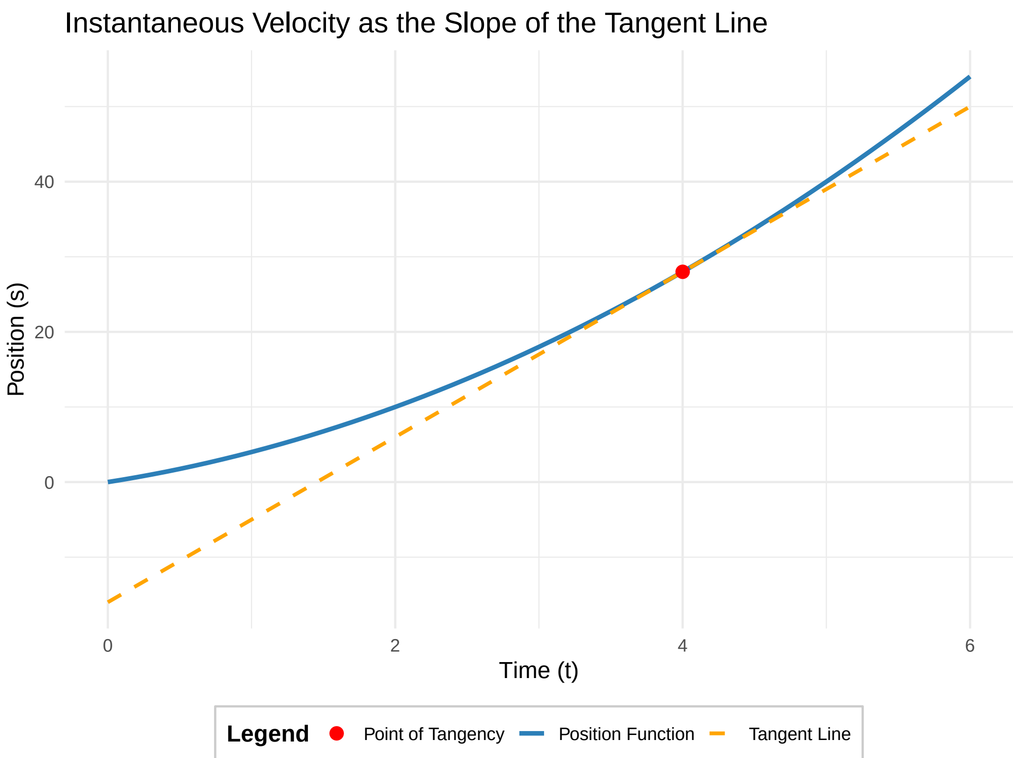

Figure 5.7: Instantaneous Velocity as the Slope of the Tangent Line

Chapter 6

Differential

In the previous chapter Limits of Functions, we built a rigorous foundation for understanding how functions behave near specific points, using the concept of limits to describe continuity and instantaneous change. This foundation naturally leads to the study of derivatives, one of the core ideas in differential calculus. In this chapter (see Figure 6.1), we introduce the derivative from a purely theoretical perspective, focusing on its formulation, interpretation as a limit, and the mathematical rules that allow us to compute derivatives efficiently. No applications are discussed here — our attention is fully on the concepts, definitions, and techniques essential for mastering differential calculus.

We explore:

- the limit-based definition of the derivative,
- alternative forms of the difference quotient,
- the geometric interpretation as the slope of a tangent line,
- rules of differentiation (power rule, product rule, quotient rule, chain rule),
- derivatives of elementary functions,
- implicit and logarithmic differentiation, = higher-order derivatives and their analytical meaning.

This chapter forms the theoretical backbone for later work involving differential equations, optimization, and applied modeling.

6.1 Concept of Derivative

The **concept of the derivative** describes how a function changes at a specific point. It provides a way to measure the **instantaneous rate of change** of a function and serves as the foundation of calculus.

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 Please view the HTML version or open directly on YouTube:
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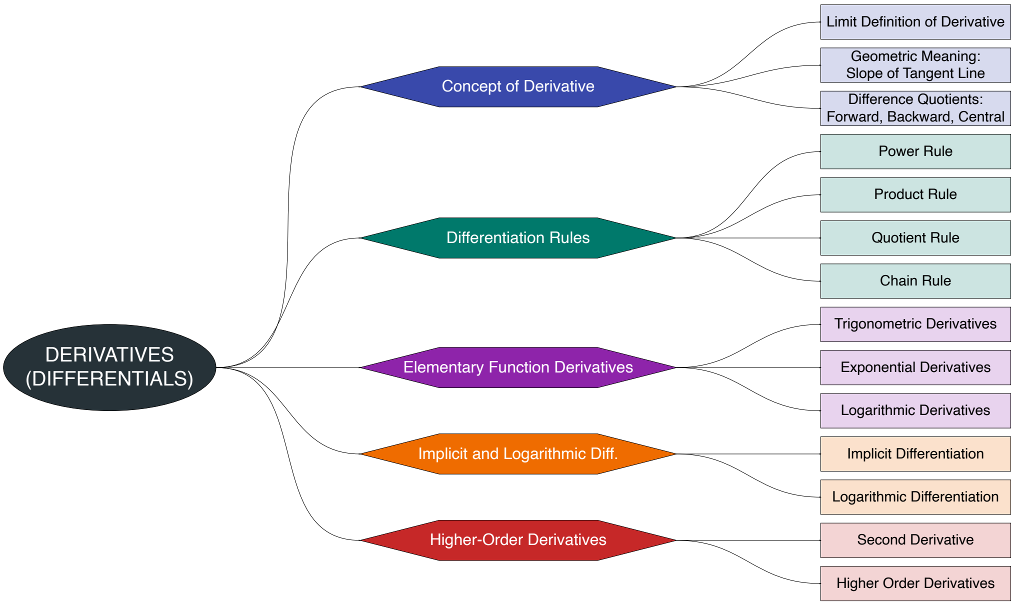


Figure 6.1: Derivatives — Concepts and Differentiation Techniques

According to the Video above, we explore the derivative through three interconnected perspectives: the limit definition $\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$, the geometric meaning of how secant lines approach the tangent line, and the role of the difference quotient in capturing the function's average rate of change before it becomes instantaneous.

Understanding the derivative begins with understanding how a function changes. Instead of looking at an entire curve at once, we first ask a simpler question: **How fast does the function change between two points?** To measure this, we compute the difference quotient:

$$\frac{f(x+h)-f(x)}{h},$$

which represents the average rate of change over the interval from x to $x+h$. Geometrically, this value corresponds to the **slope of the secant line** connecting the two points on the graph. However, the derivative is not about the average rate of change — it measures the **instantaneous rate of change** at a single point. To capture that, we let the second point move closer by making h smaller.

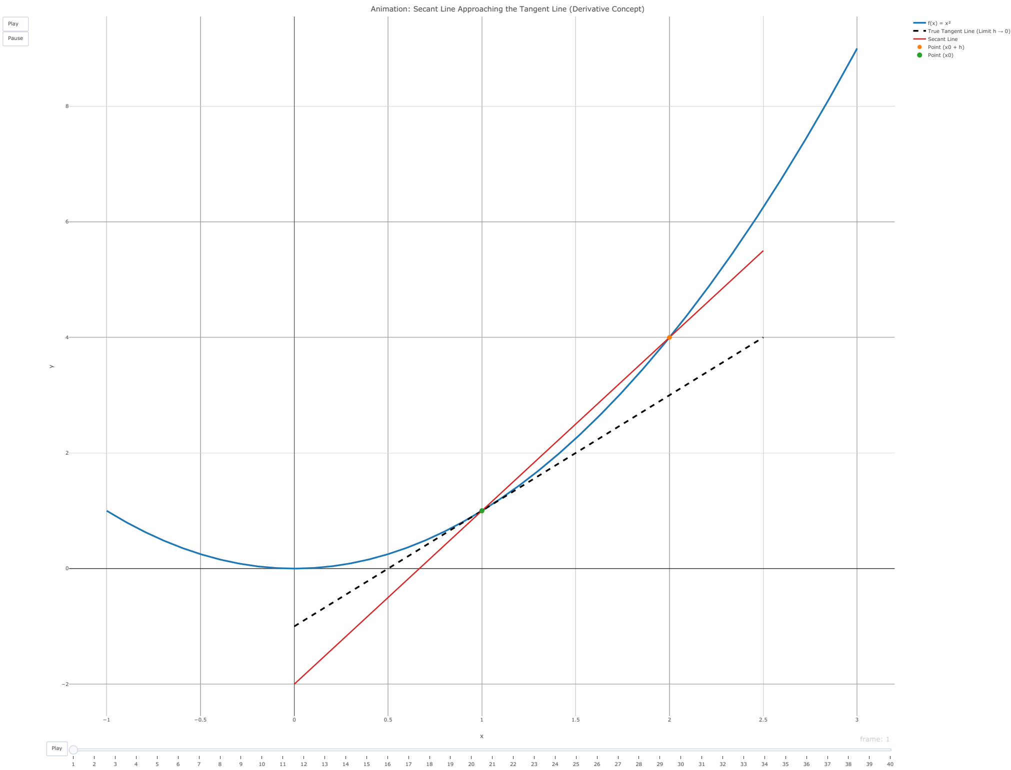
As $h \to 0$:

- the secant line begins to rotate,
- its slope changes,
- and it gradually approaches the unique line that just touches the curve at one point.

This limiting line is the **tangent line**, and its slope is defined as the derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

The following animation visually demonstrates this process. As the value of h decreases, the secant line approaches the tangent line, illustrating how the concept of a derivative emerges from the limit of the difference quotient.



Key Insights

- The secant line represents the average rate of change of the function over a finite interval h.
- As h becomes smaller, the secant line rotates and approaches the tangent line—illustrating how the derivative is defined as a limit.
- The moving point $(x_0 + h, f(x_0 + h))$ shows how the function behaves near x_0, giving an intuitive view of the idea of “approaching.”
- The tangent line in the graph represents the **instantaneous** rate of change at x_0, which is the value of the derivative.

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 Please view the HTML version or open directly on YouTube:
 https://www.youtube.com/embed/a2GJRljYhUc?si=09j3BX1z79Iylyzt
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6.2 Differentiation Rules

The following rules allow us to differentiate functions efficiently without using the limit definition.

6.2.1 Constant Rule

If c is a constant:

$$\frac{d}{dx}(c) = 0$$

6.2.2 Power Rule

For $f(x) = x^n$:

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Example ~ Differentiate:

$$f(x) = 5x^7 - 3x^3 + 2$$

Solution:

$$f'(x) = 35x^6 - 9x^2$$

6.2.3 Constant Multiple Rule

$$\frac{d}{dx}[cf(x)] = cf'(x)$$

6.2.4 Sum and Difference Rule

$$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$$

6.2.5 Product Rule

If $y = u(x)v(x)$:

$$y' = u'v + uv'$$

Example ~ Product Rule:

$$y = (3x^2 + 1)(x^3 - 4)$$

Solution:
Let $u = 3x^2 + 1$, $u' = 6x$
Let $v = x^3 - 4$, $v' = 3x^2$

$$y' = u'v + uv'$$

$$y' = 6x(x^3 - 4) + (3x^2 + 1)(3x^2)$$

$$y' = 15x^4 + 3x^2 - 24x$$

6.2.6 Quotient Rule

If $y = \frac{u}{v}$:

$$y' = \frac{u'v - uv'}{v^2}$$

Example ~ Quotient Rule

$$y = \frac{x^2 + 1}{x - 3}$$

Solution:

$$y' = \frac{2x(x-3) - (x^2 + 1)}{(x-3)^2}$$

6.2.7 Chain Rule

If $y = f(g(x))$:

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x)$$

Example 4 ~ Chain Rule:

$$f(x) = \sqrt{3x^2 + 5}$$

Rewrite:

$$f(x) = (3x^2 + 5)^{1/2}$$

$$f'(x) = \frac{1}{2}(3x^2 + 5)^{-1/2} \cdot 6x$$

$$f'(x) = \frac{3x}{\sqrt{3x^2 + 5}}$$

6.3 Elementary Function Derivatives

6.3.1 Exponential Functions

$$\frac{d}{dx}(e^x) = e^x$$
$$\frac{d}{dx}(a^x) = a^x \ln(a)$$

6.3.2 Logarithmic Functions

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$
$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln(a)}$$

Example ~Logarithmic Differentiation

$$y = (x^2+1)\sqrt{x}(3x-5)$$

Take logs:

$$\ln y = \ln(x^2+1) + \frac{1}{2}\ln x + \ln(3x-5)$$

Differentiate:

$$\frac{y'}{y} = \frac{2x}{x^2+1} + \frac{1}{2x} + \frac{3}{3x-5}$$

Thus:

$$y' = (x^2+1)\sqrt{x}(3x-5)\left(\frac{2x}{x^2+1} + \frac{1}{2x} + \frac{3}{3x-5}\right)$$

6.3.3 Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x$$
$$\frac{d}{dx}(\cos x) = -\sin x$$
$$\frac{d}{dx}(\tan x) = \sec^2 x$$

6.3.4 Inverse Trigonometric Functions

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$

6.4 Implicit Differentiation

Implicit differentiation is used when the relationship between x and y is not explicitly written as $y = f(x)$.

Example 1 ~ Implicit Differentiation:

$$x^2 + y^2 = 25$$

Differentiate both sides:

$$2x + 2y\frac{dy}{dx} = 0$$

Thus:

$$\frac{dy}{dx} = -\frac{x}{y}$$

Example 2 ~ Implicit Differentiation:

$$x^3 + y^3 = 9xy$$

Differentiate:

$$3x^2 + 3y^2y' = 9(y + xy')$$

Group terms:

$$(3y^2 - 9x)y' = 9y - 3x^2$$

Thus:

$$y' = \frac{9y - 3x^2}{3y^2 - 9x}$$

6.5 Logarithmic Differentiation

Logarithmic differentiation is especially useful for:

- variable exponents such as x^x
- products of multiple factors
- complicated quotients

Example:

$$y = x^x$$

Take the natural logarithm:

$$\ln y = x \ln x$$

Differentiate:

$$\frac{y'}{y} = \ln x + 1$$

Thus:

$$y' = x^x(\ln x + 1)$$

6.6 Higher-Order Derivatives

Higher-order derivatives are obtained by repeatedly differentiating a function.

- First derivative: y'
- Second derivative:
$$y'' = \frac{d^2y}{dx^2}$$
- Third derivative:
$$y''' = \frac{d^3y}{dx^3}$$
- n-th derivative:
$$y^{(n)}$$

Example 1 ~ Higher-Order Derivatives:
If $y = x^4$:

- $y' = 4x^3$
- $y'' = 12x^2$
- $y''' = 24x$
- $y^{(4)} = 24$

Example 2 ~ Higher-Order Derivatives:

$$y = e^{2x}$$

First derivative:

$$y' = 2e^{2x}$$

Second derivative:

$$y'' = 4e^{2x}$$

n-th derivative:

$$y^{(n)} = 2^n e^{2x}$$

References {-}

Chapter 7

Appllied of Differentials

Differentials are not merely mathematical concepts found in textbooks—they are powerful analytical tools for understanding real-world change. Through the fundamental ideas of rates of change and local approximations of functions, differentials help us predict, measure, and optimize various phenomena in everyday life as well as in professional fields [4], [5].

The video below takes us through real-world applications of differential concepts—showing how ideas about change and approximation appear in object motion, temperature variation, fluid flow, population dynamics, and various technical processes in engineering and science. By exploring these examples, we can appreciate that differentials are not merely formulas, but the foundation for precise analysis and informed decision-making..

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 https://www.youtube.com/embed/mntut9RfiIw?si=dRgVCaqK354ND1GH
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7.1 Derivatives in Metallurgy

7.1.1 Carbon Diffusion Function

We want to model the carbon concentration $C(x)$ as a function of depth x. Let us derive this systematically.

1. Observing the Data Pattern

Given data:

Depth x (mm)	Carbon (%)
0.5	0.2
1.0	0.8
1.5	1.8

To identify the type of function, examine ratios:

- Ratio of carbon increase for doubling depth:

$$\frac{C(1.0)}{C(0.5)} = \frac{0.8}{0.2} = 4$$

- Ratio of carbon increase for tripling depth:

$$\frac{C(1.5)}{C(0.5)} = \frac{1.8}{0.2} = 9$$

These ratios suggest a **power-law relationship**:

$$C(x) \propto x^n$$

where n is the exponent to be determined.

2. Determining the Exponent

Assume the general form:

$$C(x) = kx^n$$

Using the data points $(x_1, C_1) = (0.5, 0.2)$ and $(x_2, C_2) = (1.0, 0.8)$:

$$C_2/C_1 = (x_2/x_1)^n$$

Substitute the values:

$$\frac{0.8}{0.2} = \left(\frac{1.0}{0.5}\right)^n$$
$$4 = 2^n$$

Taking logarithm base 2:

$$\log_2 4 = \log_2 2^n \implies 2 = n$$

Thus, the exponent is $n = 2$, confirming a **quadratic relationship**:

$$C(x) = kx^2$$

3. Determining the Coefficient

Use any data point to solve for k, e.g., $x = 1.0$ mm, $C = 0.8$:

$$0.8 = k(1.0)^2 \implies k = 0.8$$

4. Final Model

The carbon concentration as a function of depth is:

$$\boxed{C(x) = 0.8x^2}$$

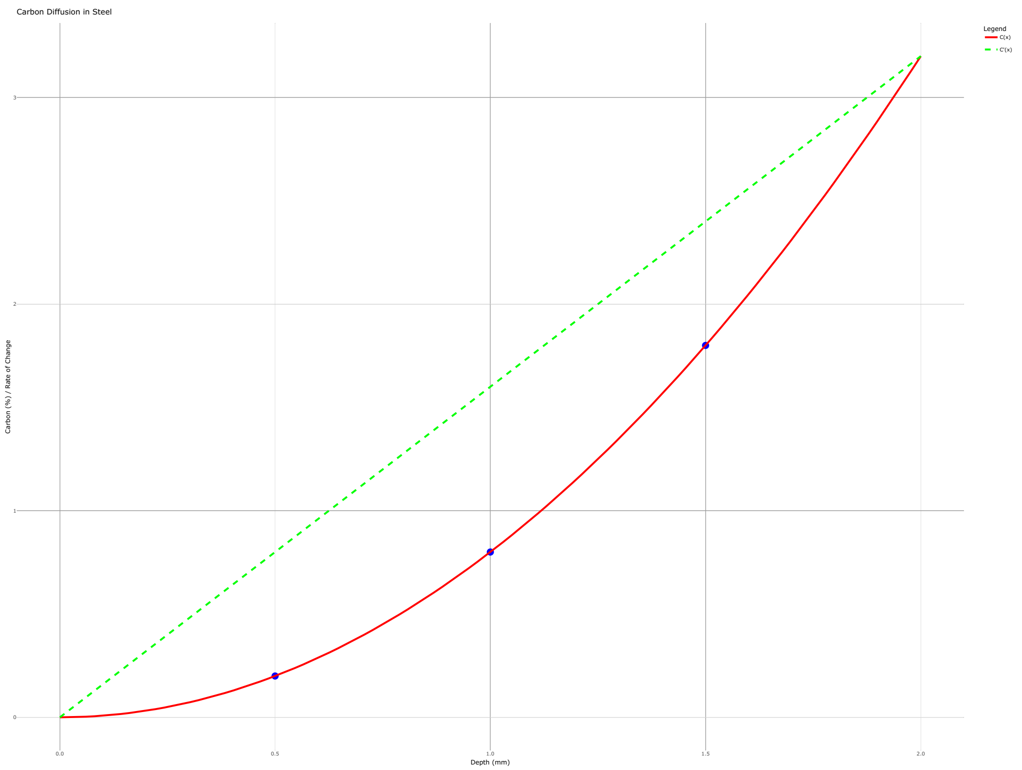
This mathematically derived function reproduces the observed quadratic increase of carbon with depth and can be used to predict concentration at other depths.

5. Derivative and Rate of Change

The rate of change is given by the derivative using the power rule:

$$C'(x) = \frac{d}{dx}[0.8x^2] = 2 \cdot 0.8 \cdot x = 1.6x$$

This derivative indicates how fast carbon concentration increases with depth, which is essential for understanding hardness and diffusion behavior.



7.1.2 Porosity Evolution

During the heating stage of a thermal treatment process (such as sintering or annealing), the porosity of a material changes due to densification, grain growth, or diffusion-driven mechanisms. The objectives of this study are to:

1. Model the evolution of porosity $P(t)$ as a function of time t and/or temperature $T(t)$.
2. Determine the rate of change (derivative) of porosity with respect to time.

3. Evaluate the suitability of simple models (linear and exponential) and examine the direct relationship between porosity and temperature.

Provided Data

Time t (min)	Porosity $P(t)$	Temperature $T(t)$ (°C)
0	0.30	800
10	0.28	830
20	0.26	860

Note: The dataset consists of three measurement points within a 0–20 minute interval, during which the temperature increases from 800 °C to 860 °C.

Density model:

$$\rho(t) = P(t)\,T(t)$$

Derivative of Pruduct Rule:

$$\rho'(t) = P'(t)T(t) + P(t)T'(t)$$

Your Task: Shows how pellet density changes as porosity decreases and temperature rises.

7.1.3 Strength vs Density

A materials laboratory is performing a heat-treatment experiment on a steel sample. During heating, the mechanical properties of the material change over time, particularly **tensile strength** and **density**.

The researchers recorded the following measurements:

Time t (min)	Strength σ (MPa)	Density ρ (g/cm³)
0	380	7.90
30	410	7.85
60	440	7.80

They want to compute the **specific strength**, defined as:

$$R(t) = \frac{\sigma(t)}{\rho(t)}$$

Use Derivative Quotient Rule:

$$R'(t) = \frac{\sigma'(t)\rho(t) - \sigma(t)\rho'(t)}{[\rho(t)]^2}$$

Your Task: Shows how efficiently strength improves compared to weight reduction.

7.1.4 Thermal Grain Growth

A materials research group studies grain growth in a metal during annealing. They report **effective annealing time** and **grain size** at three furnace temperatures:

Temperature T (°C)	Effective Time t (min)	Grain Size d (µm)
600	8	12
650	12	15
700	20	19

Grain growth during annealing is often modeled by a time-power law (at a fixed temperature):

$$d(t) = k\,t^n$$

where k and n are material- and temperature-dependent constants. Temperature itself typically influences the kinetics through an Arrhenius factor in k:

$$k(T) = k_0 \exp\left(-\frac{Q}{RT}\right),$$

with Q the activation energy, R the gas constant, and T the absolute temperature (K).

Empirical models:

- Time depends on temperature:

$$t = 2T^3$$

- Grain size depends on time:

$$d = k\,t^{1/2}$$

Derivative (Chain Rule):

$$\frac{dd}{dT} = \frac{dd}{dt} \times \frac{dt}{dT}$$

Your Task: Shows how grain size changes when annealing temperature changes.

7.1.5 Molten Metal Vibration

Data:

Time t (s)	Position x (mm)
0.00	0
0.02	4.3
0.04	3.1
0.06	-2.5

Fitted oscillation model:

$$x(t) = 5\sin(60t)$$

Derivative Trigonometric:

Velocity:

$$x'(t) = 300\cos(60t)$$

Your Task: Tells how fast molten metal is moving inside the furnace.

7.1.6 Cooling Curve of Hot Steel

Data:

Time t (s)	Temperature T (°C)
0	900
20	800
40	710
60	630

Cooling model:

$$T(t) = 900e^{-0.03t}$$

Derivative Exponential:

$$T'(t) = -27e^{-0.03t}$$

Your Task: Shows the cooling rate, which affects final microstructure.

7.1.7 Gold Leaching Concentration

Data:

Time t (h)	Gold in Solution C (mg/L)
0	0
1	27
2	36
3	41

Fitted model:

$$C(t) = \ln(1 + 4t)$$

Derivative of Logarithmic:

$$C'(t) = \frac{4}{1 + 4t}$$

Your Task: Shows how fast gold dissolves over time.

7.1.8 Ball Motion in Ball Mill

Data (Position):

Time t (s)	x (cm)
0.0	0.0
0.1	1.5
0.2	0.0
0.3	-1.5

Fitted motion model:

$$x(t) = 2\sin(10t)$$

Velocity:

$$x'(t) = 20\cos(10t)$$

Acceleration:

$$x''(t) = -200\sin(10t)$$

Your Task: Acceleration determines grinding impact energy.

7.2 Derivatives in Petroleum

7.2.1 Reservoir Pressure and Porosity

A petroleum company has collected the following reservoir data:

Depth x (m)	Pressure P (MPa)	Notes
500	15	Shallow zone
1000	25	Mid zone
1500	40	Deep zone

The **porosity** of the reservoir decreases with depth due to compaction.

7.2.2 Pressure Function

From the data, pressure increases faster than linear with depth.
We use a **power-law model**:

$$P(x) = 0.01\,x^{1.3} \quad \text{(MPa)}$$

Derivative:

$$P'(x) = \frac{d}{dx}\left(0.01x^{1.3}\right) = 0.013\,x^{0.3}$$

This indicates the **rate of pressure increase with depth**.

7.2.3 Porosity Function

Assume porosity decreases linearly with depth:

$$\phi(x) = 0.25 - 0.00005\,x$$

Derivative:

$$\phi'(x) = -0.00005$$

Indicating a **constant decrease in porosity per meter**.

7.2.4 Economic Value Function

Economic value per barrel depends on pressure:

$$V(P) = 10P^2 \quad \text{(USD per barrel)}$$

Using the chain rule:

$$\frac{dV}{dx} = \frac{dV}{dP} \cdot \frac{dP}{dx} = 20P(x)\,P'(x)$$

Substitute values:

$$\frac{dV}{dx} = 20(0.01x^{1.3})(0.013x^{0.3}) \approx 0.0026\,x^{1.6}$$

7.2.5 Total Reservoir Value per Meter

Assume fluid volume per meter is proportional to porosity:

$$T(x) = 1000\,\phi(x)$$

Total value per meter:

$$W(x) = V(P(x))\,T(x) = 1000\,P(x)^2\,\phi(x)$$

Derivative:

$$\begin{aligned} W'(x) &= 1000\left[2P(x)P'(x)\phi(x) + P(x)^2\phi'(x)\right] \\ &= 1000\left[2(0.01x^{1.3})(0.013x^{0.3})(0.25 - 0.00005x) + (0.01x^{1.3})^2(-0.00005)\right] \\ &= 0.065\,x^{1.6} - 0.00051\,x^{2.6} \end{aligned}$$

7.2.6 Reservoir Flow Analysis

Observed Data

Depth x (m)	Pressure (MPa)	Temp (°C)	Mobility (Pa·s)
500	15	60	0.12
700	20	70	0.10
1000	28	85	0.08
1300	35	100	0.06
1500	40	115	0.05

Pressure Function

Assume sinusoidal heterogeneity:

$$P(x) = 25 + 15\sin\left(\frac{\pi x}{2000}\right)$$

Temperature Function

Assume logarithmic increase:

$$T(x) = 10 + 17\ln(x)$$

Oil Mobility Function

Assume exponential decay:

$$\mu(x) = 0.15e^{-0.0007x}$$

Total Effective Flow

$$F(x) = \frac{P(x)}{\mu(x)}\cos\left(\frac{T(x)\pi}{180}\right)$$

7.3 Derivatives in Mining

7.3.1 Ore Grade and Density

Depth x (m)	Ore Grade (%)	Notes
50	0.8	Upper layer
100	1.6	Mid layer
150	3.6	Deep layer

7.3.2 Ore Grade Function

$$G(x) = 0.00637\,x^{1.2}$$

Derivative:

$$G'(x) = 0.00764\,x^{0.2}$$

7.3.3 Ore Density Function

$$\rho(x) = 2.7 - 0.002x$$

Derivative:

$$\rho'(x) = -0.002$$

7.3.4 Economic Value Function

$$V(G) = 50G^2$$

Chain rule:

$$\frac{dV}{dx} = 100\,G(x)\,G'(x) \approx 0.00486\,x^{1.4}$$

7.3.5 Total Ore Value per Meter

$$T(x) = 10\rho(x)$$

$$W(x) = 500G(x)^2\rho(x)$$

Derivative:

$$W'(x) \approx 0.1314\,x^{1.4} - 0.0405\,x^{2.4}$$

7.3.6 Copper Grade and Profit Analysis

Copper Grade Model

Model $C(x)$ using power-law (choose parameters based on regression):

$$C(x) = ax^b$$

Derivative:

$$C'(x) = abx^{b-1}$$

Extraction Cost

$$K(x) = 20 + 0.5x$$

Derivative:

$$K'(x) = 0.5$$

Profit Per Ton

$$P(x) = 100\,C(x) - K(x)$$

Derivative:

$$P'(x) = 100\,C'(x) - 0.5$$

Total Profit Per Meter

$$T(x) = 50 - 0.2x$$

$$TP(x) = P(x)T(x)$$

Derivative:

$$TP'(x) = P'(x)T(x) + P(x)T'(x)$$

7.3.7 Groundwater Flow

Water Table Height

$$H(x) = 10 + 2\sin\left(\frac{\pi x}{100}\right)$$

Derivative:

$$H'(x) = 2\frac{\pi}{100}\cos\left(\frac{\pi x}{100}\right)$$

Slope Function

$$\theta(x) = 5 + 0.5\ln(x)$$

Derivative:

$$\theta'(x) = \frac{0.5}{x}$$

Infiltration Efficiency

$$E(x) = 0.9e^{-0.02x}$$

Derivative:

$$E'(x) = -0.018e^{-0.02x}$$

Total Infiltration

$$F(x) = H(x)\,E(x)\,\cos\left(\frac{\theta(x)\pi}{180}\right)$$

7.4 References

Chapter 8

Integrals

Integration is one of the two central operations in calculus, alongside differentiation. While differentiation measures **instantaneous change**, integration measures **accumulated quantity**. In simple terms, **an integral adds up infinitely many tiny pieces** to form a total [4].

8.1 Illustrations for Integrals

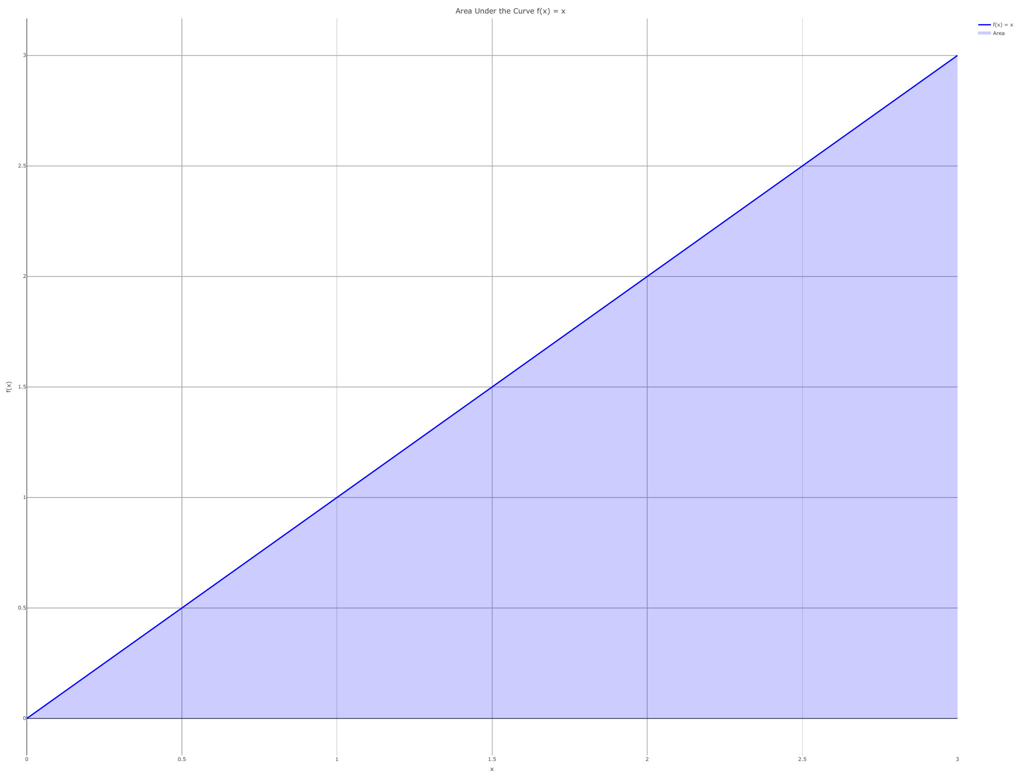
Imagine dividing a quantity into extremely small parts—lengths, areas, volumes, masses, or any measurable quantities. An integral is the limit of the sum of these tiny parts [6]. The integral symbol $\int$ was introduced by Leibniz and resembles an elongated "S", meaning "sum" [7].

Components:

- $\int$: integral (sum) symbol
- $f(x)$: function being integrated
- dx : infinitesimally small change in x [8]
- a, b : lower and upper bounds (for definite integrals)

Example:

$$\int_0^3 x.dx$$



General formula:

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$$

Apply for (n = 1):

$$\int x \, dx = \frac{x^2}{2}$$

Evaluate the definite integral:

$$\int_0^3 x \, dx = \left[\frac{x^2}{2} \right]_0^3$$

$$= \frac{3^2}{2} - \frac{0^2}{2}$$

$$= \frac{9}{2} = 4.5$$

In General a function $f(x)$, we approximate the area under the curve using rectangles of width Δx:

$$\text{Approximate Area} \approx \sum_{i=1}^{n} f(x_i)\Delta x$$

As the rectangles become thinner ($\Delta x \to 0$), the approximation becomes exact:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

This is called a **definite integral**.

8.2 Two Types of Integrals

8.2.1 Indefinite Integral

An indefinite integral represents a *family of functions* whose derivative is $f(x)$ [4] [6]:

$$\int f(x)\,dx = F(x) + C$$

where $F'(x) = f(x)$ and C is the constant of integration. This is the reverse process of differentiation [8]. Therefore, the most important link between derivatives and integrals is:

$$\frac{d}{dx}\left(\int f(t)\,dt\right) = f(x)$$

8.2.2 Definite Integral

A definite integral computes the **actual numerical value** of accumulated area or quantity between $x = a$ and $x = b$ [9]:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

It has no "$+C$" because the value is a number, not a function [10].

8.3 Area in Single Function

When we want to compute the **area** bounded by a single function $y = f(x)$ over a certain interval, we use the **definite integral** [11]. This is one of the most important ideas in calculus because it allows us to measure areas under curves that are not simple geometric shapes [12].

8.3.1 Parabola

Compute the area under:

$$f(x) = x^2$$

on the interval $0 \leq x \leq 2$.

$$A = \int_0^2 x^2 \, dx$$

Integrating:

$$\int x^2 \, dx = \frac{x^3}{3}$$

Substitute the bounds:

$$A = \left[\frac{x^3}{3} \right]_0^2 = \frac{8}{3}$$

Final Answer:

$$A = \frac{8}{3}$$

8.3.2 Linear Function

Find the area under:

$$f(x) = -x + 4$$

from $x = 0$ to $x = 4$.

$$A = \int_0^4 (-x + 4) \, dx$$

Integrate:

$$\int (-x + 4) \, dx = -\frac{x^2}{2} + 4x$$

Substitute:

$$A = \left[-\frac{x^2}{2} + 4x \right]_0^4 = 8$$

Final Answer:

$$A = 8$$

8.3.3 Function Crossing the x-axis

Consider:

$$f(x) = x - 2$$

on $[0, 4]$.
The function changes sign at $x = 2$.

Thus:

$$A = \int_0^2 |x - 2| \, dx + \int_2^4 |x - 2| \, dx$$

For $0 \leq x < 2$:

$$|x - 2| = 2 - x$$

For $2 \leq x \leq 4$:

$$|x - 2| = x - 2$$

Compute each:

$$A_1 = \int_0^2 (2 - x) \, dx = 2$$

$$A_2 = \int_2^4 (x - 2) \, dx = 2$$

Total area:

$$A = 2 + 2 = 4$$

Final Answer:

$$A = 4$$

8.3.4 Visual Explanation (Video 3)

The following video helps visualize area under a curve:

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 Please view the HTML version or open directly on YouTube:
 https://www.youtube.com/embed/IFlXeFwA_2A?si=BmSLZDshWSpojir0
```

8.4 Area in Two Function

When finding the area between two curves, we no longer measure the area under a single function. Instead, we measure the **vertical distance** between two functions across an interval.

8.4.1 Concept

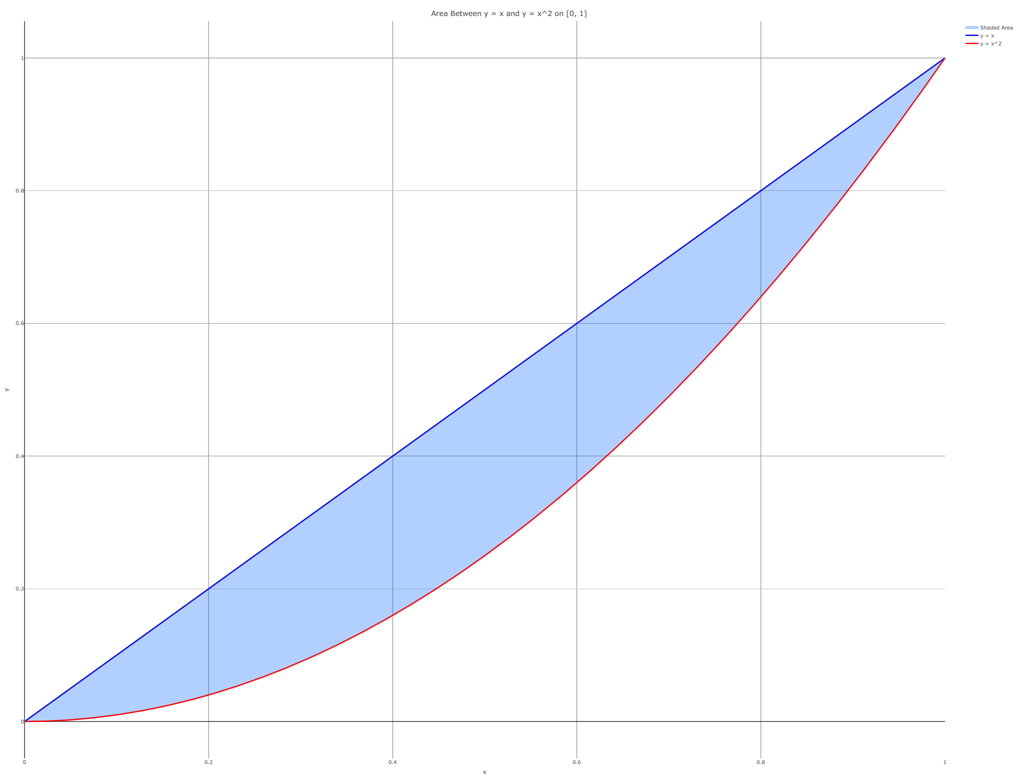
Suppose we have two functions:

- $f(x)$ — the **upper curve**
- $g(x)$ — the **lower curve**

on the interval $[a, b]$. The area between them is the accumulated difference in height:

$$\text{Area} = \int_a^b (f(x) - g(x))\ dx$$

The integrand is always **upper minus lower**, ensuring the area remains positive.



8.4.2 Steps to Compute the Area

Problem: Find the area between the curves $y = x$ and $y = x^2$ from $x = 0$ to $x = 1$.

1. **Identify which function is on top**
   For $0 \leq x \leq 1$ we have $x \geq x^2$.
   Thus $f(x) = x$ is the **upper** curve and $g(x) = x^2$ is the **lower** curve.

2. **Find the intersection points**
   Solve $f(x) = g(x)$:
   $$x = x^2 \quad \Longrightarrow \quad x(1 - x) = 0 \quad \Longrightarrow \quad x = 0, \; x = 1.$$
   These are the bounds of the region.

3. **Set up the integral**
   The area is
   $$\text{Area} = \int_0^1 (f(x) - g(x))\,dx = \int_0^1 (x - x^2)\,dx.$$

4. **Evaluate the definite integral**
   Find an antiderivative:
   $$\int (x - x^2)\,dx = \frac{x^2}{2} - \frac{x^3}{3} + C.$$

Evaluate from 0 to 1:

$$\begin{aligned}\text{Area} &= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 \\ &= \left(\frac{1^2}{2} - \frac{1^3}{3}\right) - \left(\frac{0^2}{2} - \frac{0^3}{3}\right) \\ &= \frac{1}{2} - \frac{1}{3} \\ &= \frac{3}{6} - \frac{2}{6} \\ &= \frac{1}{6}.\end{aligned}$$

Final answer:

$$\boxed{\text{Area} = \frac{1}{6}}$$

8.4.3 Visual Explanation (Video 2)

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 https://www.youtube.com/embed/fBPxOE_PmP0?si=xr1LTsfu2tsif-nu
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8.5 Volume using Integral

A solid of revolution is formed when a region in the plane is rotated around a line (axis of rotation), such as [4] [6]:

- the x-axis
- the y-axis
- any vertical or horizontal line

To compute its volume, we integrate the volume of infinitesimally thin slices [8]. In calculus, we compute the volume of a solid using definite integrals. Common methods include: **Disk Method, Washer Method,** and **Shell Method** [10] [9].

8.5.1 Disk Method

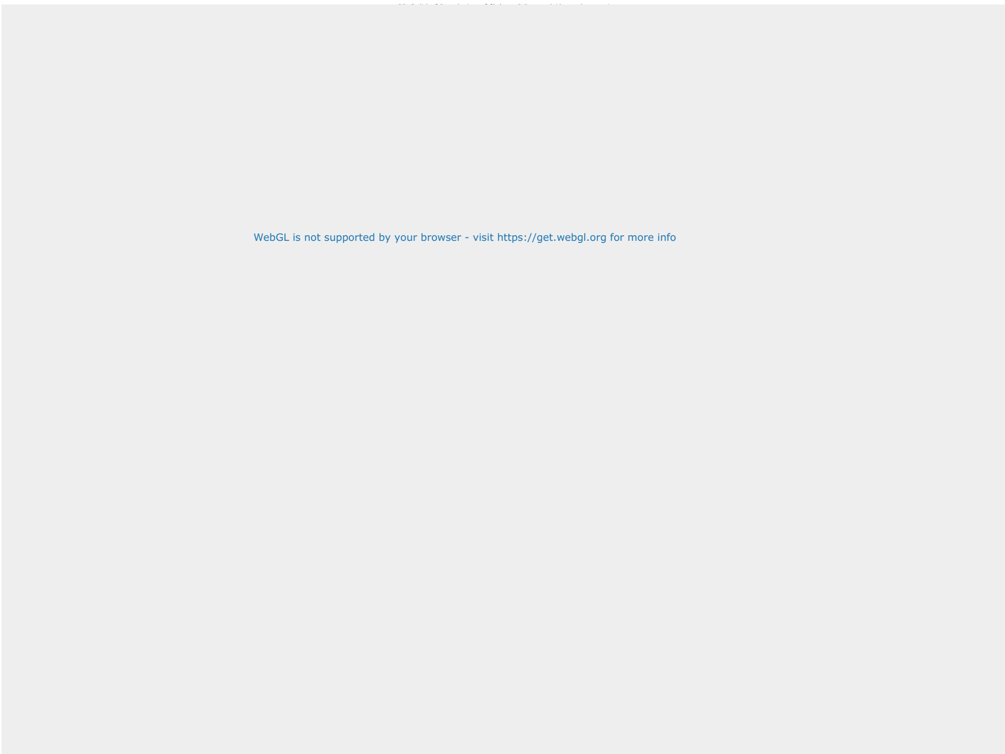
Used when the region touches the axis of rotation (no hole in the middle). If a region bounded by $y = f(x)$ is rotated around the x-axis, the volume is:

$V = \int_a^b \pi(f(x))^2 \, dx$

Each slice is a disk with:

- radius = $f(x)$
- thickness = dx

Let consider the following visualization, $f(x) = x^2$



8.5.2 Washer Method

Used when there is a gap between the region and the axis of rotation, producing a "hole". If the outer radius is $R(x)$ and the inner radius is $r(x)$, the volume is:

$V = \int_a^b \pi \left[R(x)^2 - r(x)^2 \right] \, dx$

This represents: Outer disk area minus inner disk area

8.5.3 Shell Method

Useful when rotating around the y-axis or when the function is easier to express in terms of x. If the region is rotated around the y-axis, the volume is:

$$V = \int_a^b 2\pi x \, f(x) \, dx$$

Each slice is a hollow cylindrical shell with:

- radius = x
- height = $f(x)$
- thickness = dx

8.6 Example Problem

Find the volume of the solid obtained by rotating the region bounded by $y = x^2$ and $y = 0$, for $0 \leq x \leq 1$, around the x-axis.

Solution:

Because we revolve around the x-axis, we use the **Disk Method**.

- Volume formula using disk method: $V = \pi \int_0^1 (x^2)^2 \, dx$
- Simplify the integrand: $V = \pi \int_0^1 x^4 \, dx$
- Evaluate the integral: $V = \pi \left[\frac{x^5}{5} \right]_0^1$
- Final answer: $V = \frac{\pi}{5}$

Therefore, the volume of the solid is: $\frac{\pi}{5}$

8.6.1 Visual Explanation (Video 3)

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 https://www.youtube.com/embed/DZQy2RDaSW4?si=gVJJ0bGegUZ4z83W
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References

Chapter 9

Appllied of Integrals

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 https://www.youtube.com/embed/mntut9RfiIw?si=y151tk6TLIqGQ_Ik
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9.1 Summary Applied of Integrals

Transcendental functions are those that **cannot be expressed as finite polynomials**, encompassing **exponential, logarithmic, trigonometric, and inverse trigonometric functions**. They play a pivotal role in mathematics, physics, engineering, and the applied sciences by modeling complex natural and engineered phenomena. Key types, descriptions, and example applications are summarized in Table 9.1.

References

Table 9.1: Definite Integrals

KeyConcept	Description	ExampleApplication
Definite Integral	Total accumulation of a quantity over interval $[a, b]$: $\int_a^b f(x)dx$	Area under curve: $\int_0^2 x^2 dx = \frac{8}{3}$
Area Under a Curve	Calculates area between function and x-axis	Same as above
Physical Applications	Integrals for work, mass, charge, revenue	Mass: $M = \int_a^b \rho(x)dx$, Work: $W = \int_a^b F(x)dx$

Chapter 10

Transcendental Functions

10.1 Transcendental Functions

Transcendental functions are functions that **cannot be expressed as finite polynomials**. They include **exponential, logarithmic, trigonometric, and inverse trigonometric functions**, and are essential in advanced mathematics, physics, engineering, and applied sciences for modeling complex phenomena. Transcendental functions are used to model complex phenomena in science and engineering in the table Table 10.1.

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[7] Leibniz, G. W., Historia et origo calculi differentialis, 1693

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Table 10.1: Special Functions

KeyConcept	Description	ExampleApplication
Exponential Functions	$f(x) = e^x$ or a^x, model growth and decay	RC circuit voltage: $V(t) = V_0(1 - e^{-t/RC})$
Logarithmic Functions	$f(x) = \ln x$ or $\log_a x$, used in scaling	Measuring pH, sound intensity
Trigonometric Functions	$f(x) = \sin x, \cos x, \tan x$, model periodic behavior	Wave motion: $y(x,t) = A \sin(kx - \omega t)$
Inverse Trigonometric Functions	$f(x) = \arcsin x, \arccos x, \arctan x$, solving angles	Population oscillations: $P(t) = P_{\text{avg}} + A \cos(\omega t + \phi)$

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Thank you for reading.